

VLASOV-MAXWELL-BOLTZMANN DIFFUSIVE LIMIT

JUHI JANG

ABSTRACT. We study the diffusive expansion for solutions around Maxwellian equilibrium and in a periodic box to the Vlasov-Maxwell-Boltzmann system, the most fundamental model for an ensemble of charged particles. Such an expansion yields a set of dissipative new macroscopic PDE's, the incompressible Vlasov-Navier-Stokes-Fourier system and its higher order corrections for describing a charged fluid, where the self-consistent electromagnetic field is present. The uniform estimate on the remainders is established via a unified nonlinear energy method and it guarantees the global in time validity of such an expansion up to any order.

1. INTRODUCTION AND FORMULATION

The dynamics of charged dilute particles can be described by the celebrated Vlasov-Maxwell-Boltzmann system:

$$(1.1) \quad \begin{aligned} \partial_t F_+ + v \cdot \nabla_x F_+ + (E + v \times B) \cdot \nabla_v F_+ &= Q(F_+, F_+) + Q(F_+, F_-), \\ \partial_t F_- + v \cdot \nabla_x F_- - (E + v \times B) \cdot \nabla_v F_- &= Q(F_-, F_+) + Q(F_-, F_-), \end{aligned}$$

with initial data $F_{\pm}(0, x, v) = F_{0,\pm}(x, v)$. For notational simplicity we have set all physical constants to be unity, see [8] for more background. Here $F_{\pm}(t, x, v) \geq 0$ are the spatially periodic number density functions for the ions (+) and electrons (-) respectively, at time $t \geq 0$, position $x = (x_1, x_2, x_3) \in [-\pi, \pi]^3 = \mathbb{T}^3$ and velocity $v = (v_1, v_2, v_3) \in \mathbb{R}^3$. The collision between particles is given by the standard Boltzmann collision operator $Q(G_1, G_2)$ with hard-sphere interaction:

$$(1.2) \quad Q(G_1, G_2) = \int_{\mathbb{R}^3 \times S^2} |(u - v) \cdot \omega| \{G_1(v')G_2(u') - G_1(v)G_2(u)\} dud\omega,$$

where $v' = v - [(v - u) \cdot \omega]\omega$ and $u' = u + [(v - u) \cdot \omega]\omega$.

The self-consistent, spatially periodic electromagnetic field $[E(t, x), B(t, x)]$ in (1.1) is coupled with $F_{\pm}(t, x, v)$ through the Maxwell system:

$$(1.3) \quad \begin{aligned} \partial_t E - \nabla \times B &= - \int_{\mathbb{R}^3} v(F_+ - F_-)dv, \quad \nabla \cdot B = 0, \\ \partial_t B + \nabla \times E &= 0, \quad \nabla \cdot E = \int_{\mathbb{R}^3} (F_+ - F_-)dv, \end{aligned}$$

with initial data $E(0, x) = E_0(x)$, $B(0, x) = B_0(x)$.

It turns out that it is convenient to consider the sum and difference of F_+ and F_- as proposed in [4]. Defining

$$(1.4) \quad F \equiv F_+ + F_- \text{ and } G \equiv F_+ - F_-,$$

Date: January 27, 2006.

(1.1) and (1.3) can be rewritten as following:

$$\begin{aligned}
& \partial_t F + v \cdot \nabla_x F + (E + v \times B) \cdot \nabla_v G = Q(F, F), \\
& \partial_t G + v \cdot \nabla_x G + (E + v \times B) \cdot \nabla_v F = Q(G, F), \\
(1.5) \quad & \partial_t E - \nabla \times B = - \int_{\mathbb{R}^3} v G \, dv, \quad \nabla \cdot B = 0, \\
& \partial_t B + \nabla \times E = 0, \quad \nabla \cdot E = \int_{\mathbb{R}^3} G \, dv,
\end{aligned}$$

with initial data $F(0, x, v) = F_0(x, v)$, $G(0, x, v) = G_0(x, v)$, $E(0, x) = E_0(x)$ and $B(0, x) = B_0(x)$.

Now we introduce the diffusive scaling to (1.5): for any $\varepsilon > 0$,

$$\begin{aligned}
& \varepsilon \partial_t F^\varepsilon + v \cdot \nabla_x F^\varepsilon + (E^\varepsilon + v \times B^\varepsilon) \cdot \nabla_v G^\varepsilon = \frac{1}{\varepsilon} Q(F^\varepsilon, F^\varepsilon), \\
& \varepsilon \partial_t G^\varepsilon + v \cdot \nabla_x G^\varepsilon + (E^\varepsilon + v \times B^\varepsilon) \cdot \nabla_v F^\varepsilon = \frac{1}{\varepsilon} Q(G^\varepsilon, F^\varepsilon), \\
(1.6) \quad & \varepsilon \partial_t E^\varepsilon - \nabla \times B^\varepsilon = - \int_{\mathbb{R}^3} v G^\varepsilon \, dv, \quad \nabla \cdot B^\varepsilon = 0, \\
& \varepsilon \partial_t B^\varepsilon + \nabla \times E^\varepsilon = 0, \quad \nabla \cdot E^\varepsilon = \int_{\mathbb{R}^3} G^\varepsilon \, dv.
\end{aligned}$$

For notational simplicity, we normalize the global Maxwellian as

$$(1.7) \quad \mu(v) = \frac{1}{(2\pi)^{3/2}} e^{-|v|^2/2}.$$

We consider the following formal expansion in ε around the equilibrium state $[F, G, E, B] = [\mu, 0, 0, 0]$: for any $n \geq 1$,

$$\begin{aligned}
(1.8) \quad & F^\varepsilon(t, x, v) = \mu + \sqrt{\mu} \{ \varepsilon f_1(t, x, v) + \varepsilon^2 f_2(t, x, v) + \dots + \varepsilon^n f_n^\varepsilon(t, x, v) \}, \\
& G^\varepsilon(t, x, v) = \sqrt{\mu} \{ \varepsilon g_1(t, x, v) + \varepsilon^2 g_2(t, x, v) + \dots + \varepsilon^n g_n^\varepsilon(t, x, v) \}, \\
& E^\varepsilon(t, x) = \{ \varepsilon E_1(t, x) + \varepsilon^2 E_2(t, x) + \dots + \varepsilon^n E_n^\varepsilon(t, x) \}, \\
& B^\varepsilon(t, x) = \{ \varepsilon B_1(t, x) + \varepsilon^2 B_2(t, x) + \dots + \varepsilon^n B_n^\varepsilon(t, x) \}.
\end{aligned}$$

To determine the coefficients $f_1(t, x, v), \dots, f_{n-1}(t, x, v); g_1(t, x, v), \dots, g_{n-1}(t, x, v); E_1(t, x), \dots, E_{n-1}(t, x); B_1(t, x), \dots, B_{n-1}(t, x)$, we plug the formal diffusive expansion (1.8) into the rescaled equations (1.6):

$$\begin{aligned}
(1.9) \quad & (\varepsilon \partial_t + v \cdot \nabla_x) \{ \varepsilon f_1 + \dots + \varepsilon^n f_n^\varepsilon \} \\
& + \frac{1}{\sqrt{\mu}} \{ \varepsilon (E_1 + v \times B_1) + \dots + \varepsilon^n (E_n^\varepsilon + v \times B_n^\varepsilon) \} \cdot \nabla_v [\sqrt{\mu} \{ \varepsilon g_1 + \dots + \varepsilon^n g_n^\varepsilon \}] \\
& = \frac{1}{\varepsilon \sqrt{\mu}} Q(\mu + \sqrt{\mu} \{ \varepsilon f_1 + \dots + \varepsilon^n f_n^\varepsilon \}, \mu + \sqrt{\mu} \{ \varepsilon f_1 + \dots + \varepsilon^n f_n^\varepsilon \}), \\
& (\varepsilon \partial_t + v \cdot \nabla_x) \{ \varepsilon g_1 + \dots + \varepsilon^n g_n^\varepsilon \} \\
& + \frac{1}{\sqrt{\mu}} \{ \varepsilon (E_1 + v \times B_1) + \dots + \varepsilon^n (E_n^\varepsilon + v \times B_n^\varepsilon) \} \cdot \nabla_v [\mu + \sqrt{\mu} \{ \varepsilon f_1 + \dots + \varepsilon^n f_n^\varepsilon \}] \\
& = \frac{1}{\varepsilon \sqrt{\mu}} Q(\sqrt{\mu} \{ \varepsilon g_1 + \dots + \varepsilon^n g_n^\varepsilon \}, \mu + \sqrt{\mu} \{ \varepsilon f_1 + \dots + \varepsilon^n f_n^\varepsilon \}),
\end{aligned}$$

$$\begin{aligned}
\varepsilon \partial_t \{\varepsilon E_1 + \dots + \varepsilon^n E_n^\varepsilon\} - \nabla \times \{\varepsilon B_1 + \dots + \varepsilon^n B_n^\varepsilon\} &= - \int_{\mathbb{R}^3} v \sqrt{\mu} \{\varepsilon g_1 + \dots + \varepsilon^n g_n^\varepsilon\} dv, \\
\varepsilon \partial_t \{\varepsilon B_1 + \dots + \varepsilon^n B_n^\varepsilon\} + \nabla \times \{\varepsilon E_1 + \dots + \varepsilon^n E_n^\varepsilon\} &= 0, \\
\nabla \cdot \{\varepsilon E_1 + \dots + \varepsilon^n E_n^\varepsilon\} &= \int_{\mathbb{R}^3} \sqrt{\mu} \{\varepsilon g_1 + \dots + \varepsilon^n g_n^\varepsilon\} dv, \\
\nabla \cdot \{\varepsilon B_1 + \dots + \varepsilon^n B_n^\varepsilon\} &= 0.
\end{aligned}$$

To expand the right hand side Q in the above, we define L the well-known linearized collision operator and \mathcal{L} another linearized operator as

$$(1.10) \quad Lf \equiv -\frac{1}{\sqrt{\mu}} \{Q(\mu, \sqrt{\mu}f) + Q(\sqrt{\mu}f, \mu)\},$$

$$(1.11) \quad \mathcal{L}g \equiv -\frac{1}{\sqrt{\mu}} Q(\sqrt{\mu}g, \mu),$$

and the nonlinear collision operator Γ as (non-symmetric)

$$(1.12) \quad \Gamma(f, g) \equiv \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu}f, \sqrt{\mu}g).$$

Note that Lf and $\mathcal{L}g$ can be written as following in terms of Γ :

$$(1.13) \quad Lf = -\{\Gamma(\sqrt{\mu}, f) + \Gamma(f, \sqrt{\mu})\}, \quad \mathcal{L}g = -\Gamma(g, \sqrt{\mu}).$$

Now we equate the coefficients on both sides of the equation (1.9) in front of different powers of the parameter ε . Let

$$f_{-1} = f_0 \equiv 0, \quad g_{-1} = g_0 \equiv 0, \quad E_0 \equiv 0, \quad B_0 \equiv 0$$

to obtain

$$\begin{aligned}
&\partial_t f_m + v \cdot \nabla_x f_{m+1} + \frac{1}{\sqrt{\mu}} \sum_{\substack{i+j=m+1 \\ i,j \geq 1}} (E_i + v \times B_i) \cdot \nabla_v (\sqrt{\mu} g_j) \\
&= -Lf_{m+2} + \sum_{\substack{i+j=m+2 \\ i,j \geq 1}} \Gamma(f_i, f_j), \\
(1.14) \quad &\partial_t g_m + v \cdot \nabla_x g_{m+1} + \frac{1}{\sqrt{\mu}} \sum_{\substack{i+j=m+1 \\ i,j \geq 1}} (E_i + v \times B_i) \cdot \nabla_v (\sqrt{\mu} f_j) - E_{m+1} \cdot v \sqrt{\mu} \\
&= -\mathcal{L}g_{m+2} + \sum_{\substack{i+j=m+2 \\ i,j \geq 1}} \Gamma(g_i, f_j),
\end{aligned}$$

for $-1 \leq m \leq n-3$ as well as

$$\begin{aligned}
(1.15) \quad &\partial_t E_m - \nabla \times B_{m+1} = - \int v g_{m+1} \sqrt{\mu} dv, \quad \nabla \cdot B_{m+1} = 0, \\
&\partial_t B_m + \nabla \times E_{m+1} = 0, \quad \nabla \cdot E_{m+1} = \int g_{m+1} \sqrt{\mu} dv,
\end{aligned}$$

for $0 \leq m \leq n-2$. Moreover, we can collect terms left in (1.9) with powers ε^{n-1} or higher and divide by ε^{n-1} to get the equations for the remainders $f_n^\varepsilon, g_n^\varepsilon, E_n^\varepsilon, B_n^\varepsilon$.

Note that all the ε^{m+1} -th order terms vanish for $m \leq n-3$ because of (1.14). First, we write equations for f_n^ε and g_n^ε :

(1.16)

$$\begin{aligned}
& \varepsilon^2 \partial_t f_n^\varepsilon + \varepsilon v \cdot \nabla_x f_n^\varepsilon + L f_n^\varepsilon = \\
& \{-\partial_t f_{n-2} - v \cdot \nabla_x f_{n-1} - \frac{1}{\sqrt{\mu}} \sum_{\substack{i+j=n-1 \\ i,j \geq 1}} (E_i + v \times B_i) \cdot \nabla_v (\sqrt{\mu} g_j) + \sum_{\substack{i+j=n \\ i,j \geq 1}} \Gamma(f_i, f_j)\} \\
& + \varepsilon \{-\partial_t f_{n-1} - \frac{1}{\sqrt{\mu}} \sum_{\substack{i+j=n \\ i,j \geq 1}} (E_i + v \times B_i) \cdot \nabla_v (\sqrt{\mu} g_j) + \sum_{\substack{i+j=n+1 \\ i,j \geq 1}} \Gamma(f_i, f_j)\} \\
& + \varepsilon^n \Gamma(f_n^\varepsilon, f_n^\varepsilon) + \sum_{i=1}^{n-1} \varepsilon^i \{\Gamma(f_n^\varepsilon, f_i) + \Gamma(f_i, f_n^\varepsilon)\} + \sum_{i+j \geq n+2} \varepsilon^{i+j-n} \Gamma(f_i, f_j) \\
& - \frac{\varepsilon^{n+1}}{\sqrt{\mu}} (E_n^\varepsilon + v \times B_n^\varepsilon) \cdot \nabla_v (\sqrt{\mu} g_n^\varepsilon) \\
& - \frac{1}{\sqrt{\mu}} \sum_{i=1}^{n-1} \varepsilon^{i+1} \{(E_i + v \times B_i) \cdot \nabla_v (\sqrt{\mu} g_n^\varepsilon) + (E_n^\varepsilon + v \times B_n^\varepsilon) \cdot \nabla_v (\sqrt{\mu} g_i)\} \\
& - \frac{1}{\sqrt{\mu}} \sum_{i+j \geq n+1} \varepsilon^{i+j-n+1} (E_i + v \times B_i) \cdot \nabla_v (\sqrt{\mu} g_j);
\end{aligned}$$

$$\begin{aligned}
& \varepsilon^2 \partial_t g_n^\varepsilon + \varepsilon v \cdot \nabla_x g_n^\varepsilon - \varepsilon E_n^\varepsilon \cdot v \sqrt{\mu} + \mathcal{L} g_n^\varepsilon = \\
& \{-\partial_t g_{n-2} - v \cdot \nabla_x g_{n-1} - \frac{1}{\sqrt{\mu}} \sum_{\substack{i+j=n-1 \\ i,j \geq 1}} (E_i + v \times B_i) \cdot \nabla_v (\sqrt{\mu} f_j) + E_{n-1} \cdot v \sqrt{\mu} \\
& + \sum_{\substack{i+j=n \\ i,j \geq 1}} \Gamma(g_i, f_j)\} + \varepsilon \{-\partial_t g_{n-1} - \frac{1}{\sqrt{\mu}} \sum_{\substack{i+j=n \\ i,j \geq 1}} (E_i + v \times B_i) \cdot \nabla_v (\sqrt{\mu} f_j) \\
& + \sum_{\substack{i+j=n+1 \\ i,j \geq 1}} \Gamma(g_i, f_j)\} + \varepsilon^n \Gamma(g_n^\varepsilon, f_n^\varepsilon) + \sum_{i=1}^{n-1} \varepsilon^i \{\Gamma(g_n^\varepsilon, f_i) + \Gamma(g_i, f_n^\varepsilon)\} \\
& + \sum_{i+j \geq n+2} \varepsilon^{i+j-n} \Gamma(g_i, f_j) - \frac{\varepsilon^{n+1}}{\sqrt{\mu}} (E_n^\varepsilon + v \times B_n^\varepsilon) \cdot \nabla_v (\sqrt{\mu} f_n^\varepsilon) \\
& - \frac{1}{\sqrt{\mu}} \sum_{i=1}^{n-1} \varepsilon^{i+1} \{(E_i + v \times B_i) \cdot \nabla_v (\sqrt{\mu} f_n^\varepsilon) + (E_n^\varepsilon + v \times B_n^\varepsilon) \cdot \nabla_v (\sqrt{\mu} f_i)\} \\
& - \frac{1}{\sqrt{\mu}} \sum_{i+j \geq n+1} \varepsilon^{i+j-n+1} (E_i + v \times B_i) \cdot \nabla_v (\sqrt{\mu} f_j).
\end{aligned}$$

Similarly, by (1.15), we get the remainders for $E_n^\varepsilon, B_n^\varepsilon$:

$$\begin{aligned}
(1.17) \quad & \varepsilon \partial_t E_n^\varepsilon - \nabla \times B_n^\varepsilon = -\partial_t E_{n-1} - \int_{\mathbb{R}^3} v g_n^\varepsilon \sqrt{\mu} dv, \quad \nabla \cdot B_n^\varepsilon = 0, \\
& \varepsilon \partial_t B_n^\varepsilon + \nabla \times E_n^\varepsilon = -\partial_t B_{n-1}, \quad \nabla \cdot E_n^\varepsilon = \int_{\mathbb{R}^3} g_n^\varepsilon \sqrt{\mu} dv.
\end{aligned}$$

The fluid equations can be obtained through the conditions (1.14) and (1.15). We first recall that the operator $L \geq 0$, and for any fixed (t, x) , the null space of L is generated by $[\sqrt{\mu}, v\sqrt{\mu}, |v|^2\sqrt{\mu}]$. For any function $f(t, x, v)$ we thus can decompose

$$f = \mathbf{P}_1 f + \{\mathbf{I} - \mathbf{P}_1\}f$$

where $\mathbf{P}_1 f$ (hydrodynamic part) is the L_v^2 projection on the null space for L for given (t, x) . We can further denote

$$(1.18) \quad \mathbf{P}_1 f = \{\rho_f(t, x) + v \cdot u_f(t, x) + (\frac{|v|^2}{2} - \frac{3}{2})\theta_f(t, x)\}\sqrt{\mu}.$$

Here we define the *hydrodynamic field* of f to be

$$[\rho_f(t, x), u_f(t, x), \theta_f(t, x)]$$

which represents the density, velocity and temperature fluctuations physically. For the velocity field $u_f(t, x)$, we further define its *divergent-free* part as

$$P_0 u_f = \text{the divergent-free projection of } u_f$$

so that

$$\nabla \cdot \{P_0 u_f\} \equiv 0.$$

Similarly, one can show that $\mathcal{L} \geq 0$ and for any fixed (t, x) , the null space of \mathcal{L} is one dimensional vector space generated by $[\sqrt{\mu}]$. Likewise, any $g(t, x, v)$ can be decomposed into

$$g = \mathbf{P}_2 g + \{\mathbf{I} - \mathbf{P}_2\}g$$

where $\mathbf{P}_2 g$ (hydrodynamic part) is the L_v^2 projection on the null space for \mathcal{L} for given (t, x) . We can further denote

$$(1.19) \quad \mathbf{P}_2 g = \sigma_g(t, x)\sqrt{\mu}.$$

Here $\sigma_g(t, x)$, the *hydrodynamic field* of g , can be interpreted as the concentration difference. For more details about \mathcal{L} and \mathbf{P}_2 , we refer [4]. Before going on, we state the coercivity of L and \mathcal{L} which will be often used in the subsequent sections: there exists a $\delta > 0$ such that

$$(1.20) \quad \langle Lf, f \rangle \geq \delta |(\mathbf{I} - \mathbf{P}_1)f|_\nu^2, \quad \langle \mathcal{L}g, g \rangle \geq \delta |(\mathbf{I} - \mathbf{P}_2)g|_\nu^2.$$

See Lemma 1 in [8] for its proof. Note that the operator L defined in [8] is equivalent to $[L, \mathcal{L}]$ in our case.

Now define $[\rho_m, u_m, \theta_m, \sigma_m]$ to be the corresponding hydrodynamic field of the m -th coefficients f_m and g_m . As for the first coefficients $f_1(t, x, v)$ and $g_1(t, x, v)$, from (1.14)

$$(1.21) \quad \{\mathbf{I} - \mathbf{P}_1\}f_1 = 0 \quad \text{and} \quad \{\mathbf{I} - \mathbf{P}_2\}g_1 = 0$$

which immediately yield that

$$(1.22) \quad B_1 = 0 \quad \text{and} \quad E_1 = \nabla \phi_1$$

up to constant and for some function $\phi_1(t, x)$ satisfying $\Delta \phi_1 = \sigma_1$; in particular, B_1 may be assumed to be zero physically in a sense that nonzero constants B_1 do not cause the hydrodynamic equations (1.25)-(1.28) to change. It will be shown in Lemma 3.1 that its velocity fluctuation $u_1(t, x)$ is incompressible:

$$(1.23) \quad \nabla \cdot u_1 \equiv 0 \quad \text{or} \quad u_1 = P_0 u_1,$$

and its density and temperature fluctuations $\rho_1(t, x)$ and $\theta_1(t, x)$ satisfy the Boussinesq relation:

$$(1.24) \quad \rho_1 + \theta_1 \equiv 0.$$

Moreover, $[u_1, \theta_1, \sigma_1]$ satisfies the nonlinear incompressible Vlasov-Navier-Stokes-Fourier equations:

$$(1.25) \quad \partial_t u_1 + u_1 \cdot \nabla u_1 + \nabla p_1 = \eta \Delta u_1 + \sigma_1 \nabla \phi_1,$$

$$(1.26) \quad \partial_t \sigma_1 + u_1 \cdot \nabla \sigma_1 = \alpha \Delta \sigma_1 - \alpha \sigma_1,$$

$$(1.27) \quad \Delta \phi_1 = \sigma_1,$$

$$(1.28) \quad \partial_t \theta_1 + u_1 \cdot \nabla \theta_1 = \kappa \Delta \theta_1,$$

where $p_1(t, x)$ is the pressure and $\eta, \kappa, \alpha > 0$ are physical constants.

As for the coefficients $f_m(t, x, v)$, $g_m(t, x, v)$ for $m \geq 2$, by (1.14), the microscopic part of f_m and g_m is determined by:

$$(1.29) \quad \begin{aligned} \{\mathbf{I} - \mathbf{P}_1\} f_m &= L^{-1} \{ -\partial_t f_{m-2} - v \cdot \nabla_x f_{m-1} + \sum_{\substack{i+j=m \\ i,j \geq 1}} \Gamma(f_i, f_j) \\ &\quad - \frac{1}{\sqrt{\mu}} \sum_{\substack{i+j=m-1 \\ i,j \geq 1}} (E_i + v \times B_i) \cdot \nabla_v (\sqrt{\mu} g_j) \}, \\ \{\mathbf{I} - \mathbf{P}_2\} g_m &= \mathcal{L}^{-1} \{ -\partial_t g_{m-2} - v \cdot \nabla_x g_{m-1} + \sum_{\substack{i+j=m \\ i,j \geq 1}} \Gamma(g_i, f_j) \\ &\quad - \frac{1}{\sqrt{\mu}} \sum_{\substack{i+j=m-1 \\ i,j \geq 1}} (E_i + v \times B_i) \cdot \nabla_v (\sqrt{\mu} f_j) + E_{m-1} \cdot v \sqrt{\mu} \}. \end{aligned}$$

On the other hand, for the hydrodynamic field of f_m and g_m :

$$\begin{aligned} \mathbf{P}_1 f_m &= \{ \rho_m(t, x) + v \cdot u_m(t, x) + \{ \frac{|v|^2}{2} - \frac{3}{2} \} \theta_m(t, x) \} \sqrt{\mu}, \\ \mathbf{P}_2 g_m &= \sigma_m(t, x) \sqrt{\mu}, \end{aligned}$$

we can deduce an m -th order incompressibility condition

$$(1.31) \quad \nabla \cdot \{I - P_0\} u_m = -\partial_t \rho_{m-1},$$

an m -th order Boussinesq relation

$$(1.32) \quad \begin{aligned} \rho_m + \theta_m &= \Delta^{-1} \nabla \cdot \{ -u_1 \cdot \nabla (P_0 u_{m-1}) - P_0 u_{m-1} \cdot \nabla u_1 + E_1 \sigma_{m-1} + E_{m-1} \sigma_1 \\ &\quad + R_{m-1}^u \} + \langle \frac{|v|^2 \sqrt{\mu}}{3}, L^{-1}(\{\mathbf{I} - \mathbf{P}_1\} v \cdot \nabla_x \mathbf{P}_1 f_{m-1}) \rangle - \frac{5}{2} \theta_1 \theta_{m-1} - u_{m-1} \cdot u_1, \end{aligned}$$

and an m -th order *linear* Vlasov-Navier-Stokes-Fourier system for $[P_0 u_m, \theta_m, \sigma_m]$:

$$(1.33) \quad (\partial_t + u_1 \cdot \nabla - \eta \Delta) P_0 u_m + P_0 u_m \cdot \nabla u_1 + \nabla p_m - (E_1 \sigma_m + E_m \sigma_1) = R_m^u,$$

$$(1.34) \quad (\partial_t + u_1 \cdot \nabla - \alpha \Delta + \alpha) \sigma_m + P_0 u_m \cdot \nabla \sigma_1 = R_m^\sigma,$$

$$(1.35) \quad (\partial_t + u_1 \cdot \nabla - \kappa \Delta) \theta_m + P_0 u_m \cdot \nabla \theta_1 = R_m^\theta,$$

$$(1.36) \quad \nabla \times E_m = -\partial_t B_{m-1}, \quad \nabla \cdot E_m = \sigma_m,$$

$$(1.37) \quad \nabla \times B_m = \int_{\mathbb{R}^3} \{\mathbf{I} - \mathbf{P}_2\} g_m v \sqrt{\mu} dv + \partial_t E_{m-1}, \quad \nabla \cdot B_m = 0,$$

with compatibility conditions coming from conservation laws

$$(1.38) \quad \frac{d}{dt} \int_{\mathbb{T}^3} E_m dx = -\alpha \int_{\mathbb{T}^3} E_m dx + \ell_{m-1}, \quad \int_{\mathbb{T}^3} B_m dx = 0.$$

Here $R_m^u, R_m^\sigma, R_m^\theta$ and ℓ_{m-1} , defined precisely in (3.2), (3.3), (3.4) and (3.20), essentially depend only on f_j, g_j, E_j, B_j for $j \leq m-1$, since $\{\mathbf{I} - \mathbf{P}_1\}f_m, \{\mathbf{I} - \mathbf{P}_2\}g_m, \{I - P_0\}u_m$, as well as $\rho_m + \theta_m$ have been determined.

In order to state our results precisely in the next section, we introduce the following norms and notations. We use $\langle \cdot, \cdot \rangle$ to denote the standard L^2 inner product in \mathbb{R}^3 , while we use (\cdot, \cdot) to denote L^2 inner product either in $\mathbb{T}^3 \times \mathbb{R}^3$ or in \mathbb{T}^3 with corresponding L^2 norm $\|\cdot\|$. We use the standard notation H^s to denote the Sobolev space $W^{s,2}$. For the Boltzmann collision operator (1.2), define the collision frequency to be

$$(1.39) \quad \nu(v) \equiv \int_{\mathbb{R}^3} |v - v'| \mu(v') dv',$$

which behaves like $|v|$ as $|v| \rightarrow \infty$. It is natural to define the following weighted L^2 norm to characterize the dissipation rate.

$$|g|_\nu^2 \equiv \int_{\mathbb{R}^3} g^2(v) \nu(v) dv, \quad \|g\|_\nu^2 \equiv \int_{\mathbb{T}^3 \times \mathbb{R}^3} g^2(x, v) \nu(v) dv dx.$$

Observe that for hard sphere interaction,

$$(1.40) \quad \|(1 + |v|)^{\frac{1}{2}} g\| \leq C \|g\|_\nu.$$

In order to be consistent with the hydrodynamic equations, we define

$$(1.41) \quad \partial_\gamma^\beta = \partial_{x_1}^{\gamma_1} \partial_{x_2}^{\gamma_2} \partial_{x_3}^{\gamma_3} \partial_{v_1}^{\beta_1} \partial_{v_2}^{\beta_2} \partial_{v_3}^{\beta_3}$$

where $\gamma = [\gamma_1, \gamma_2, \gamma_3]$ is related to the space derivatives, while $\beta = [\beta_1, \beta_2, \beta_3]$ is related to the velocity derivatives.

We now define instant energy functionals and the dissipation rate.

Definition 1 (Instant Energy) For $N \geq 8$, for some constant $C > 0$, an instant energy functional $\mathcal{E}_N(f, g, E, B)(t) \equiv \mathcal{E}_N(t)$ satisfies:

$$(1.42) \quad \frac{1}{C} \mathcal{E}_N(t) \leq \sum_{|\beta|+|\gamma| \leq N} \|[\partial_\gamma^\beta f, \partial_\gamma^\beta g]\|^2(t) + \sum_{|\gamma| \leq N} \|[\partial_\gamma E, \partial_\gamma B]\|^2(t) \leq C \mathcal{E}_N(t).$$

Definition 2 (Dissipation Rate) For $N \geq 8$, the dissipation rate $\mathcal{D}_N(f, g)(t)$ is defined as

$$(1.43) \quad \begin{aligned} \mathcal{D}_N(f, g)(t) &= \sum_{|\gamma| \leq N} \|[\partial_\gamma \mathbf{P}_1 f, \partial_\gamma \mathbf{P}_2 g]\|^2(t) \\ &+ \frac{1}{\varepsilon^2} \sum_{|\beta|+|\gamma| \leq N} \|[\partial_\gamma^\beta \{\mathbf{I} - \mathbf{P}_1\} f, \partial_\gamma^\beta \{\mathbf{I} - \mathbf{P}_2\} g]\|_\nu^2(t). \end{aligned}$$

We remark that both the instant energy and the dissipation rate are carefully designed to capture the structure of the rescaled Vlasov-Maxwell-Boltzmann equation

(1.6). First of all, the electromagnetic field $[E, B]$ is included only in the instant energy, which prevents the exponential decay on \mathcal{E}_N unlike the pure Boltzmann case for hard potentials. See [9] and [16]. Notice that there is no $\frac{1}{\varepsilon^2}$ factor in front of the hydrodynamic part $[\mathbf{P}_1 f, \mathbf{P}_2 g]$ in the dissipation rate $\mathcal{D}_N(f, g)$, since only the microscopic part $[\{\mathbf{I} - \mathbf{P}_1\}f, \{\mathbf{I} - \mathbf{P}_2\}g]$ should vanish as $\varepsilon \rightarrow 0$. For notational simplicity, the *Einstein's summation convention* is used for Greek letter up to order $N \geq 8$ throughout the paper, unless otherwise specified. We denote $\nabla = \nabla_x$ and use C to denote a constant independent of ε . We also use $U(\cdot)$ to denote a general positive polynomial with $U(0) = 0$.

2. MAIN RESULTS

The first result is to determine the coefficients $f_1, f_2, \dots, f_m; g_1, g_2, \dots, g_m; E_1, E_2, \dots, E_m; B_1, B_2, \dots, B_m$ in a diffusive approximation (1.8).

Theorem 2.1. *Let m divergent-free vector-valued functions $[u_1^0(x), u_2^0(x), \dots, u_m^0(x)]$, $2m$ scalar functions $[\theta_1^0(x), \theta_2^0(x), \dots, \theta_m^0(x); \sigma_1^0(x), \sigma_2^0(x), \dots, \sigma_m^0(x)]$ be given such that*

$$\|u_1^0\|_{H^2} + \|\theta_1^0\|_{H^2} + \|\sigma_1^0\|_{H^2} \leq M$$

and

$$\begin{aligned} \int_{\mathbb{T}^3} \sigma_r^0(x) dx &= 0, \quad \int_{\mathbb{T}^3} u_r^0(x) dx = - \sum_{\substack{i+j=r \\ i,j \geq 1}} \int_{\mathbb{T}^3} E_i^0(x) \times B_j^0(x) dx, \\ \int_{\mathbb{T}^3} \frac{3}{2} \theta_r^0(x) dx &= -\frac{1}{2} \sum_{\substack{i+j=r \\ i,j \geq 1}} \int_{\mathbb{T}^3} E_i^0(x) \cdot E_j^0(x) + B_i^0(x) \cdot B_j^0(x) dx, \end{aligned}$$

for $1 \leq r \leq m$. Here E_i^0, B_j^0 are defined by $E_i(0, x), B_j(0, x)$ which have been inductively determined at the precedents since $i, j < r$, starting with the average conditions $\int_{\mathbb{T}^3} B_1^0 dx = 0$ and $\int_{\mathbb{T}^3} E_1^0 dx = 0$. Then for sufficiently small M and given m real vectors $e_1(=0), e_2, \dots, e_m$, there exist unique functions $f_1(t, x, v), f_2(t, x, v), \dots, f_m(t, x, v); g_1(t, x, v), g_2(t, x, v), \dots, g_m(t, x, v); E_1(t, x), E_2(t, x), \dots, E_m(t, x)$ and $B_1(t, x), B_2(t, x), \dots, B_m(t, x)$ with

$$\begin{aligned} \int_{\mathbb{T}^3} \sigma_r(t, x) dx &= 0, \quad \int_{\mathbb{T}^3} P_0 u_r(t, x) dx = - \sum_{\substack{i+j=r \\ i,j \geq 1}} \int_{\mathbb{T}^3} E_i(t, x) \times B_j(t, x) dx, \\ (2.1) \quad \int_{\mathbb{T}^3} \frac{3}{2} \theta_r(t, x) dx &= -\frac{1}{2} \sum_{\substack{i+j=r \\ i,j \geq 1}} \int_{\mathbb{T}^3} E_i(t, x) \cdot E_j(t, x) + B_i(t, x) \cdot B_j(t, x) dx, \end{aligned}$$

such that initially $P_0 u_r(0, x) = u_r^0(x)$, $\theta_r(0, x) = \theta_r^0(x)$, $\sigma_r(0, x) = \sigma_r^0(x)$ and $\int_{\mathbb{T}^3} E_r(0, x) dx = e_r$, and $f_1(t, x, v), g_1(t, x, v), E_1(t, x), B_1(t, x)$ satisfy (1.21)-(1.28) and $f_r(t, x, v), g_r(t, x, v), E_r(t, x), B_r(t, x)$ satisfy (1.29)-(1.38) for $2 \leq r \leq m$.

Moreover, for $1 \leq r \leq m$, for any β , and for all $s \geq 3$, there exists a polynomial $U_{r,\beta,s}$ with $U_{r,\beta,s}(0) = 0$ such that

$$(2.2) \quad \sum_{|\tau| \leq s} \{ ||[\partial_\tau^\beta f_r, \partial_\tau^\beta g_r]||_\nu(t) + ||[\partial_\tau E_r, \partial_\tau B_r]||_1(t) \} \\ \leq e^{-\lambda t} U_{r,\beta,s} \left(\sum_{1 \leq j \leq r} \{ ||u_j^0||_{H^{2s+4(r-j)}} + ||\theta_j^0||_{H^{2s+4(r-j)}} + ||\sigma_j^0||_{H^{2s+4(r-j)}} \} \right),$$

where space-time derivatives

$$\partial_\tau = \partial_t^{\tau_0} \partial_{x_1}^{\tau_1} \partial_{x_2}^{\tau_2} \partial_{x_3}^{\tau_3}$$

and λ can be chosen as $\frac{1}{4} \min\{\eta, \kappa, \alpha\}$ for sufficiently small M .

We now turn to the most important question about the remainder estimates for f_n^ε , g_n^ε , E_n^ε and B_n^ε . We first study the classical case for the first order remainders

$$f^\varepsilon \equiv f_1^\varepsilon, \quad g^\varepsilon \equiv g_1^\varepsilon, \quad E^\varepsilon \equiv E_1^\varepsilon, \quad B^\varepsilon \equiv B_1^\varepsilon$$

which satisfy the nonlinear Boltzmann type equations:

$$(2.3) \quad \partial_t f^\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f^\varepsilon + \frac{1}{\varepsilon^2} L f^\varepsilon = \frac{1}{\varepsilon} \Gamma(f^\varepsilon, f^\varepsilon) - \frac{1}{\sqrt{\mu}} (E^\varepsilon + v \times B^\varepsilon) \cdot \nabla_v (\sqrt{\mu} g^\varepsilon), \\ \partial_t g^\varepsilon + \frac{1}{\varepsilon} v \cdot (\nabla_x g^\varepsilon - \sqrt{\mu} E^\varepsilon) + \frac{1}{\varepsilon^2} \mathcal{L} g^\varepsilon = \frac{1}{\varepsilon} \Gamma(g^\varepsilon, f^\varepsilon) - \frac{1}{\sqrt{\mu}} (E^\varepsilon + v \times B^\varepsilon) \cdot \nabla_v (\sqrt{\mu} f^\varepsilon),$$

$$(2.4) \quad \varepsilon \partial_t E^\varepsilon - \nabla \times B^\varepsilon = - \int g^\varepsilon v \sqrt{\mu} dv, \quad \nabla \cdot B^\varepsilon = 0, \\ \varepsilon \partial_t B^\varepsilon + \nabla \times E^\varepsilon = 0, \quad \nabla \cdot E^\varepsilon = \int g^\varepsilon \sqrt{\mu} dv.$$

Theorem 2.2. *Let $N \geq 8$. Let $f^\varepsilon(0, x, v) = f_0^\varepsilon(x, v)$, $g^\varepsilon(0, x, v) = g_0^\varepsilon(x, v)$ and $E^\varepsilon(0, x) = E_0^\varepsilon(x)$, $B^\varepsilon(0, x) = B_0^\varepsilon(x)$ satisfy the mass, momentum and energy conservation laws:*

$$(2.5) \quad (f_0^\varepsilon, \sqrt{\mu}) = 0, \quad (g_0^\varepsilon, \sqrt{\mu}) = 0, \\ (f_0^\varepsilon, v \sqrt{\mu}) + \varepsilon \int_{\mathbb{T}^3} E_0^\varepsilon \times B_0^\varepsilon dx = 0, \\ (f_0^\varepsilon, |v|^2 \sqrt{\mu}) + \varepsilon \int_{\mathbb{T}^3} |E_0^\varepsilon|^2 + |B_0^\varepsilon|^2 dx = 0.$$

Then there exists an instant energy functional $\mathcal{E}_N(f^\varepsilon, g^\varepsilon, E^\varepsilon, B^\varepsilon)(t)$ such that if $\mathcal{E}_N(f^\varepsilon, g^\varepsilon, E^\varepsilon, B^\varepsilon)(0)$ is sufficiently small, then

$$(2.6) \quad \frac{d}{dt} \mathcal{E}_N(f^\varepsilon, g^\varepsilon, E^\varepsilon, B^\varepsilon)(t) + \mathcal{D}_N(f^\varepsilon, g^\varepsilon)(t) \leq 0.$$

In particular,

$$(2.7) \quad \sup_{0 \leq t \leq \infty} \mathcal{E}_N(f^\varepsilon, g^\varepsilon, E^\varepsilon, B^\varepsilon)(t) \leq \mathcal{E}_N(f^\varepsilon, g^\varepsilon, E^\varepsilon, B^\varepsilon)(0).$$

Moreover, for $k \geq 1$ there exists $C_{N,k} > 0$ such that

$$(2.8) \quad \mathcal{E}_N(f^\varepsilon, g^\varepsilon, E^\varepsilon, B^\varepsilon)(t) \leq C_{N,k} \mathcal{E}_{N+k}(f^\varepsilon, g^\varepsilon, E^\varepsilon, B^\varepsilon)(0) \left\{ 1 + \frac{t}{k} \right\}^{-k}.$$

We remark that our initial data $f_0^\varepsilon, g_0^\varepsilon, E_0^\varepsilon, B_0^\varepsilon$ are general and can contain initial layer for the Vlasov-Navier-Stokes-Fourier limit. We can easily include (one) temporal derivative ∂_t in our definition of instant energy and dissipation rate, and obtain the same uniform bound for such a new norm. With such a modification, the boundedness of $\partial_t f_0^\varepsilon, \partial_t g_0^\varepsilon, \partial_t E_0^\varepsilon, \partial_t B_0^\varepsilon$ automatically removes the formation of any initial layer.

For higher order remainders $f_n^\varepsilon, g_n^\varepsilon, E_n^\varepsilon, B_n^\varepsilon$ with $n \geq 2$, we have

Theorem 2.3. *Let $N \geq 8$. Given $f_1, f_2, \dots, f_n; g_1, g_2, \dots, g_n; E_1, E_2, \dots, E_n; B_1, B_2, \dots, B_n$ constructed in Theorem 2.1 and let*

$$(2.9) \quad \begin{aligned} & |||[\mathbf{u}_n^0, \theta_n^0, \sigma_n^0]|||_N(t) \\ & \equiv \sum_{1 \leq j \leq n} \{ |||u_i^0|||_{H^{2N+10+4(n-j)}} + |||\theta_i^0|||_{H^{2N+10+4(n-j)}} + |||\sigma_i^0|||_{H^{2N+10+4(n-j)}} \}. \end{aligned}$$

And let

$$(2.10) \quad \begin{aligned} F^\varepsilon(0, x, v) & \equiv \mu + \sqrt{\mu} \{ \varepsilon f_1(0, x, v) + \dots + \varepsilon^{n-1} f_{n-1}(0, x, v) + \varepsilon^n f_n^\varepsilon(0, x, v) \}, \\ G^\varepsilon(0, x, v) & \equiv \sqrt{\mu} \{ \varepsilon g_1(0, x, v) + \dots + \varepsilon^{n-1} g_{n-1}(0, x, v) + \varepsilon^n g_n^\varepsilon(0, x, v) \}, \\ E^\varepsilon(0, x) & \equiv \varepsilon E_1(0, x) + \dots + \varepsilon^{n-1} E_{n-1}(0, x) + \varepsilon^n E_n^\varepsilon(0, x), \\ B^\varepsilon(0, x) & \equiv \varepsilon B_1(0, x) + \dots + \varepsilon^{n-1} B_{n-1}(0, x) + \varepsilon^n B_n^\varepsilon(0, x) \end{aligned}$$

be given initial data satisfying the following conservation laws:

$$(2.11) \quad \begin{aligned} & \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \{ F^\varepsilon(0, x, v) - \mu(v) \} dv dx = 0, \quad \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} G^\varepsilon(0, x, v) dv dx = 0, \\ & \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} v F^\varepsilon(0, x, v) dv dx + \int_{\mathbb{T}^3} E^\varepsilon(0, x) \times B^\varepsilon(0, x) dx = 0, \\ & \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |v|^2 \{ F^\varepsilon(0, x, v) - \mu(v) \} dv dx + \int_{\mathbb{T}^3} |E^\varepsilon(0, x)|^2 + |B^\varepsilon(0, x)|^2 dx = 0. \end{aligned}$$

Then there exist an instant energy functional \mathcal{E}_N and a positive polynomial U with $U(0) = 0$ such that if both ε and

$$\mathcal{E}_N(f_n^\varepsilon - f_n, g_n^\varepsilon - g_n, E_n^\varepsilon - E_n, B_n^\varepsilon - B_n)(0)$$

are sufficiently small, then

$$(2.12) \quad \begin{aligned} & \sup_{0 \leq t \leq \infty} \mathcal{E}_N(f_n^\varepsilon - f_n, g_n^\varepsilon - g_n, E_n^\varepsilon - E_n, B_n^\varepsilon - B_n)(t) \\ & \leq \{ e^{U(|||[\mathbf{u}_n^0, \theta_n^0, \sigma_n^0]|||_N)} \mathcal{E}_N(f_n^\varepsilon - f_n, g_n^\varepsilon - g_n, E_n^\varepsilon - E_n, B_n^\varepsilon - B_n)(0) \\ & \quad + \varepsilon^2 U(|||[\mathbf{u}_n^0, \theta_n^0, \sigma_n^0]|||_N) \}. \end{aligned}$$

Moreover, for $k \geq 1$ there exists $C_{N,k} > 0$ such that

$$(2.13) \quad \begin{aligned} & \mathcal{E}_N(f_n^\varepsilon - f_n, g_n^\varepsilon - g_n, E_n^\varepsilon - E_n, B_n^\varepsilon - B_n)(t) \\ & \leq C_{N,k} \{ e^{U(|||[\mathbf{u}_n^0, \theta_n^0, \sigma_n^0]|||_N)} \mathcal{E}_{N+k}(f_n^\varepsilon - f_n, g_n^\varepsilon - g_n, E_n^\varepsilon - E_n, B_n^\varepsilon - B_n)(0) \\ & \quad + \varepsilon^2 U(|||[\mathbf{u}_n^0, \theta_n^0, \sigma_n^0]|||_N) \} \left\{ 1 + \frac{t}{k} \right\}^{-k}. \end{aligned}$$

Note that the conservation laws (2.11) in Theorem 2.3 imply the conservation laws for $f_n^\varepsilon, g_n^\varepsilon, E_n^\varepsilon, B_n^\varepsilon$ due to (2.1). See (7.5) for the precise description. In addition, we remark that the positivity for the initial data $(F^\varepsilon \pm G^\varepsilon)(0, x, v) \geq 0$

which is equivalent to $F_{\pm}^{\varepsilon}(0, x, v) \geq 0$ through the relation (1.4) can be verified in the same way as discussed in the Appendix of [9].

There are several macroscopic fluid models for classifying the dynamics of a charged fluid, but none has been derived from the Boltzmann theory mathematically. This is because the construction of the global solution to the important Vlasov-Maxwell-Boltzmann system had been open for a long time until only a few years ago, in [8], a unique global in time classical solution near a global Maxwellian for such a master system was constructed. However, it still remains a major open problem: to construct global renormalized solutions to the same system.

Singular limit problems emanating from the Boltzmann equations have been studied by many people for decades [1, 2, 3, 5, 6, 7, 9, 10, 11, 12, 14, 15, 17]—a great overview of the issue is given in [9, 13, 18]. In particular, in the recent work [9], higher order approximations with the unified energy method have now been shown to give rise to a rigorous passage from the Boltzmann equation to the Navier-Stokes-Fourier systems beyond the Navier-Stokes approximation.

In this article, we rigorously establish the global in time validity of the diffusive expansion (1.8) to the rescaled Vlasov-Maxwell-Boltzmann equations (1.6) for any order. This not only gives the uniform estimates for the higher order corrections, but also leads to the mathematical derivation of new dissipative hydrodynamic equations, which we call Vlasov-Navier-Stokes-Fourier System. An interesting feature of our result is that for the chosen incompressible regime, the nontrivial magnetic effect in the limit first occurs in the second order expansion ($n = 2$), while the electric effect is important at any order. We believe our result opens a new line of research for those macroscopic approximations both physically and mathematically.

The method of this paper is based on the improvement of the recently developed nonlinear energy method in [8, 9]. We use the reformulation (1.5), by introducing new unknowns (1.4), to simplify both the character of the Vlasov-Maxwell-Boltzmann system and the analysis presented in this paper endowed with the clear, concrete linearized collision operators L and \mathcal{L} . Macroscopic equations and local conservation laws are proven to be key tools to build the crucial positivity of L and \mathcal{L} . The most important analytical difficulty lies in deriving the uniform estimates on f^{ε} and g^{ε} . Due to the singular behavior of the time derivative in our problem, the positivity estimate for purely spatial derivatives is invoked: see (5.4) in Lemma 5.1. Notice that an additional term $\frac{d}{dt}G(t)$ is needed. Such a differential form can still yield decay estimates with the notion of equivalent instant energies. The local conservation laws are used to estimate the more singular temporal derivative for the hydrodynamic field directly in terms of purely spatial derivatives of the microscopic part. We also use the trick of the integration by parts in the time variable and, in turn, by using such a uniform estimate, avoid encountering $[\partial_t\{\mathbf{I} - \mathbf{P}_1\}f^{\varepsilon}, \partial_t\{\mathbf{I} - \mathbf{P}_2\}g^{\varepsilon}]$. This complication is another reason for introducing the notion of equivalent instant energies.

As pointed out in [8], the Vlasov-Maxwell-Boltzmann system is tricky to handle due to the hyperbolic nature of Maxwell equation and indeed, it is the most intriguing and critical part of this article to control electromagnetic fields. For that, we first utilize the macroscopic equation (5.13) to estimate electric fields and then the Maxwell equation itself for magnetic fields. Noting that the macroscopic equation (5.13) is simpler than the one considered in [8] owing to the reformulation (1.5),

we remark that our method together with the positivity estimate (5.4) provides another lucid and concise way of proving the global in time classical solution for the Vlasov-Maxwell-Boltzmann equations, especially without using the temporal derivatives.

On the other hand, it is delicate to establish the well-posedness of the new hydrodynamic equations because of their complexity, mainly stemming from electromagnetic fields. It turns out that the compatibility conditions (1.38) for averages of electromagnetic fields are necessary and sufficient conditions in order for the Vlasov-Navier-Stokes-Fourier systems to have the unique solution. We should point out that those conditions are natural restrictions in that they can be derived from conservation laws. In turn, the exponential decay of hydrodynamic variables as well as the solvability of hydrodynamic equations lead to the solvability of kinetic equations and almost the same decay for approximate solutions. While the same exponential decay rate is obtained for the pure Boltzmann case, we are able to obtain only the polynomial decay rate (2.8) and (2.13) for solutions of the Vlasov-Maxwell-Boltzmann equations (compare with (2.6) and (2.13) in [9]) by applying the method proposed in [16]. In particular, for higher order remainders the more sophisticated continuity argument is employed in order to get the desired polynomial decay rate (2.13) from (7.15).

The paper will proceed as follows: we will derive the high order linear Vlasov-Navier-Stokes-Fourier system in Section 3; we will prove Theorem 2.1 for coefficients $f_1(t, x, v), \dots, f_m(t, x, v), g_1(t, x, v), \dots, g_m(t, x, v), E_1(t, x), \dots, E_m(t, x), B_1(t, x), \dots, B_m(t, x)$ in Section 4; Section 5 will be devoted to the positivity of L and \mathcal{L} ; in the last two sections, the first and higher order remainder estimates—Theorem 2.2 and Theorem 2.3—will be proven respectively.

3. HIGH ORDER VLASOV-NAVIER-STOKES-FOURIER SYSTEM: FORMAL DERIVATION

In this section, we derive the microscopic equations (1.29), (1.30) and hydrodynamic equations (1.31)-(1.38). We shall use many results from [9].

Lemma 3.1. *Assume that the expansion (1.8) satisfies (1.6) and such that for $|\tau| + |\beta| \leq N$,*

$$(3.1) \quad \sum_{m=1}^{n-1} \{ ||[\partial_\tau^\beta f_m, \partial_\tau^\beta g_m]||_\nu + ||[\partial_\tau E_m, \partial_\tau B_m]|| \} \\ + ||[\partial_\tau^\beta f_n^\varepsilon, \partial_\tau^\beta g_n^\varepsilon]||_\nu + ||[\partial_\tau E_n^\varepsilon, \partial_\tau B_n^\varepsilon]|| < \infty.$$

Then there exist $f_n(t, x, v), g_n(t, x, v), f_{n+1}(t, x, v), g_{n+1}(t, x, v)$ and $E_n(t, x), B_n(t, x)$ such that (1.14) and (1.15) are valid for all $m \leq n-1$. Moreover, f_1 and g_1 satisfy (1.21), the incompressibility condition (1.23), the Boussinesq relation (1.24), and the first order Vlasov-Navier-Stokes-Fourier system (1.25)-(1.28); E_1 and B_1 satisfy (1.22). For $m \geq 2$, f_m, g_m, E_m, B_m satisfy the microscopic equation (1.29) and (1.30), the m -th order incompressibility condition (1.31), the m -th order Boussinesq relation (1.32), and the m -th order Vlasov-Navier-Stokes-Fourier system (1.33)-(1.38) with

$$\begin{aligned}
R_m^u &\equiv \langle v \cdot \nabla_x L^{-1} \{ \partial_t \{ \mathbf{I} - \mathbf{P}_1 \} f_{m-1} - \sum_{\substack{i+j=m+1, \\ i,j \geq 1}} \Gamma(f_i, f_j) \}, v \sqrt{\mu} \rangle \\
&+ \langle v \cdot \nabla_x L^{-1} \{ \{ \mathbf{I} - \mathbf{P}_1 \} (v \cdot \nabla_x \{ \mathbf{I} - \mathbf{P}_1 \} f_m), v \sqrt{\mu} \rangle \\
&- \langle v \cdot \nabla_x L^{-1} \{ \Gamma(f_1, \{ \mathbf{I} - \mathbf{P}_1 \} f_m) + \Gamma(\{ \mathbf{I} - \mathbf{P}_1 \} f_m, f_1) \}, v \sqrt{\mu} \rangle \\
(3.2) \quad &+ \langle v \cdot \nabla_x L^{-1} \{ \frac{1}{\sqrt{\mu}} \sum_{\substack{i+j=m, \\ i,j \geq 1}} (E_i + v \times B_i) \cdot \nabla_v (g_j \sqrt{\mu}) \}, v \sqrt{\mu} \rangle \\
&- (\partial_t + u_1 \cdot \nabla - \eta \Delta) \{ I - P_0 \} u_m - \{ I - P_0 \} u_m \cdot \nabla u_1 \\
&- (\nabla \cdot \{ I - P_0 \} u_m) u_1 + \frac{\eta}{3} \nabla (\nabla \cdot \{ I - P_0 \} u_m) \\
&+ \sum_{\substack{i+j=m+1, \\ i,j \geq 1}} \{ E_i \nabla \cdot E_j - (\partial_t E_{j-1} - \nabla \times B_j) \times B_i \};
\end{aligned}$$

$$\begin{aligned}
R_m^\sigma &\equiv \langle v \cdot \nabla_x \mathcal{L}^{-1} \{ \partial_t \{ \mathbf{I} - \mathbf{P}_2 \} g_{m-1} - \sum_{\substack{i+j=m+1, \\ i,j \geq 1}} \Gamma(g_i, f_j) \}, \sqrt{\mu} \rangle \\
&+ \langle v \cdot \nabla_x \mathcal{L}^{-1} \{ \{ \mathbf{I} - \mathbf{P}_2 \} (v \cdot \nabla_x \{ \mathbf{I} - \mathbf{P}_2 \} g_m), \sqrt{\mu} \rangle \\
(3.3) \quad &- \langle v \cdot \nabla_x \mathcal{L}^{-1} \{ \Gamma(g_1, \{ \mathbf{I} - \mathbf{P}_2 \} g_m) + \Gamma(\{ \mathbf{I} - \mathbf{P}_2 \} g_m, g_1) \}, \sqrt{\mu} \rangle \\
&+ \langle v \cdot \nabla_x \mathcal{L}^{-1} \{ \frac{1}{\sqrt{\mu}} \sum_{\substack{i+j=m, \\ i,j \geq 1}} (E_i + v \times B_i) \cdot \nabla_v (f_j \sqrt{\mu}) \}, \sqrt{\mu} \rangle \\
&- \{ \nabla \cdot (I - P_0) u_m \} \sigma_1 - (I - P_0) u_m \cdot \nabla \sigma_1;
\end{aligned}$$

$$\begin{aligned}
R_m^\theta &\equiv \langle v \cdot \nabla_x L^{-1} \{ \partial_t \{ \mathbf{I} - \mathbf{P}_1 \} f_{m-1} - \sum_{\substack{i+j=m+1, \\ i,j \geq 1}} \Gamma(f_i, f_j) \}, \frac{|v|^2 \sqrt{\mu}}{5} \rangle \\
&+ \langle v \cdot \nabla_x L^{-1} \{ \{ \mathbf{I} - \mathbf{P}_1 \} (v \cdot \nabla_x \{ \mathbf{I} - \mathbf{P}_1 \} f_m), \frac{|v|^2 \sqrt{\mu}}{5} \rangle \\
&- \langle v \cdot \nabla_x L^{-1} \{ \Gamma(f_1, \{ \mathbf{I} - \mathbf{P}_1 \} f_m) + \Gamma(\{ \mathbf{I} - \mathbf{P}_1 \} f_m, f_1) \}, \frac{|v|^2 \sqrt{\mu}}{5} \rangle \\
(3.4) \quad &+ \langle v \cdot \nabla_x L^{-1} \{ \frac{1}{\sqrt{\mu}} \sum_{\substack{i+j=m, \\ i,j \geq 1}} (E_i + v \times B_i) \cdot \nabla_v (g_j \sqrt{\mu}) \}, \frac{|v|^2 \sqrt{\mu}}{5} \rangle \\
&+ (\frac{2}{5} \partial_t - u_1 \cdot \nabla + \kappa_1 \Delta) \{ \rho_m + \theta_m \} \\
&- \{ \nabla \cdot (I - P_0) u_m \} \theta_1 - (I - P_0) u_m \cdot \nabla \theta_1 \\
&- \frac{2}{5} \sum_{\substack{i+j=m+1, \\ i,j \geq 1}} E_i \cdot (\partial_t E_{j-1} - \nabla \times B_j);
\end{aligned}$$

$$(3.5) \quad p_m \equiv (\rho_{m+1} + \theta_{m+1}) - \left\langle \frac{|v|^2 \sqrt{\mu}}{3}, L^{-1} \{ \{\mathbf{I} - \mathbf{P}_1\} (v \cdot \nabla_x \mathbf{P}_1 f_m) \} \right\rangle \\ + \frac{5}{2} \theta_1 \theta_m + (u_m \cdot u_1).$$

Also the conservation laws (2.1) are valid for each m .

Proof. Notice that (1.14) is clearly valid for $m < n - 2$ under the assumption (3.1) since it can be derived by letting $\varepsilon \rightarrow 0$ after dividing (1.9) by ε^{m+1} . By the same token, (1.15) holds for $m < n - 1$. We now show the existence of the coefficients f_n, g_n, E_n, B_n and f_{n+1}, g_{n+1} so that the equations (1.14) for $m = n - 2, n - 1$ and (1.15) for $m = n - 1$ are satisfied. Firstly, by (3.1), up to a subsequence, there exist f_n, g_n, E_n, B_n such that

$$[f_n^\varepsilon, g_n^\varepsilon] \rightharpoonup [f_n, g_n] \text{ weakly in } \|\cdot\|_\nu \text{ and } [E_n^\varepsilon, B_n^\varepsilon] \rightharpoonup [E_n, B_n] \text{ weakly in } \|\cdot\|.$$

The assumption (3.1) guarantees the existence of their derivatives. By subtracting off $Lf_n, \mathcal{L}g_n$ on both sides of (1.16) we can isolate the zeroth order terms in the remainder equations and deduce (1.14) for $m = n - 2$ at least in the sense of distributions. Moreover, letting $\varepsilon \rightarrow 0$ of (1.17), we also deduce (1.15) for $m = n - 1$ at least in the sense of distributions.

Next, in order to find f_{n+1} and g_{n+1} , we use (1.14) for $m = n - 2$ to simplify (1.16) as

$$(3.6) \quad L \left\{ \frac{f_n^\varepsilon - f_n}{\varepsilon} \right\} \\ = -\varepsilon \partial_t f_n^\varepsilon - v \cdot \nabla_x f_n^\varepsilon \\ + \left\{ -\partial_t f_{n-1} - \frac{1}{\sqrt{\mu}} \sum_{\substack{i+j=n \\ i,j \geq 1}} (E_i + v \times B_i) \cdot \nabla_v (\sqrt{\mu} g_j) + \sum_{\substack{i+j=n+1 \\ i,j \geq 1}} \Gamma(f_i, f_j) \right\} \\ + \varepsilon^{n-1} \Gamma(f_n^\varepsilon, f_n^\varepsilon) + \sum_{i=1}^{n-1} \varepsilon^{i-1} \{ \Gamma(f_n^\varepsilon, f_i) + \Gamma(f_i, f_n^\varepsilon) \} + \sum_{i+j \geq n+2} \varepsilon^{i+j-n-1} \Gamma(f_i, f_j) \\ - \frac{\varepsilon^n}{\sqrt{\mu}} (E_n^\varepsilon + v \times B_n^\varepsilon) \cdot \nabla_v (\sqrt{\mu} g_n^\varepsilon) \\ - \frac{1}{\sqrt{\mu}} \sum_{i=1}^{n-1} \varepsilon^i \{ (E_i + v \times B_i) \cdot \nabla_v (\sqrt{\mu} g_n^\varepsilon) + (E_n^\varepsilon + v \times B_n^\varepsilon) \cdot \nabla_v (\sqrt{\mu} g_i) \} \\ - \frac{1}{\sqrt{\mu}} \sum_{i+j \geq n+1} \varepsilon^{i+j-n} (E_i + v \times B_i) \cdot \nabla_v (\sqrt{\mu} g_j);$$

$$\begin{aligned}
& \mathcal{L} \left\{ \frac{g_n^\varepsilon - g_n}{\varepsilon} \right\} \\
&= -\varepsilon \partial_t g_n^\varepsilon - v \cdot \nabla_x g_n^\varepsilon + E_n^\varepsilon \cdot v \sqrt{\mu} \\
&+ \left\{ -\partial_t g_{n-1} - \frac{1}{\sqrt{\mu}} \sum_{\substack{i+j=n \\ i,j \geq 1}} (E_i + v \times B_i) \cdot \nabla_v (\sqrt{\mu} f_j) + \sum_{\substack{i+j=n+1 \\ i,j \geq 1}} \Gamma(g_i, f_j) \right\} \\
(3.7) \quad &+ \varepsilon^{n-1} \Gamma(g_n^\varepsilon, f_n^\varepsilon) + \sum_{i=1}^{n-1} \varepsilon^{i-1} \{ \Gamma(g_n^\varepsilon, f_i) + \Gamma(g_i, f_n^\varepsilon) \} \\
&+ \sum_{i+j \geq n+2} \varepsilon^{i+j-n-1} \Gamma(g_i, f_j) - \frac{\varepsilon^n}{\sqrt{\mu}} (E_n^\varepsilon + v \times B_n^\varepsilon) \cdot \nabla_v (\sqrt{\mu} f_n^\varepsilon) \\
&- \frac{1}{\sqrt{\mu}} \sum_{i=1}^{n-1} \varepsilon^i \{ (E_i + v \times B_i) \cdot \nabla_v (\sqrt{\mu} f_n^\varepsilon) + (E_n^\varepsilon + v \times B_n^\varepsilon) \cdot \nabla_v (\sqrt{\mu} f_i) \} \\
&- \frac{1}{\sqrt{\mu}} \sum_{i+j \geq n+1} \varepsilon^{i+j-n} (E_i + v \times B_i) \cdot \nabla_v (\sqrt{\mu} f_j).
\end{aligned}$$

Take the inner product of (3.6) with $\{\mathbf{I} - \mathbf{P}_1\} \left\{ \frac{f_n^\varepsilon - f_n}{\varepsilon} \right\}$. By (1.20), the LHS of (3.6) is bounded from below by

$$\delta \|(\mathbf{I} - \mathbf{P}_1) \left\{ \frac{f_n^\varepsilon - f_n}{\varepsilon} \right\}\|_\nu^2.$$

On the other hand, for the inner products in the RHS of (3.6), by (1.40), we have

$$\begin{aligned}
& (-\varepsilon \partial_t f_n^\varepsilon - v \cdot \nabla_x f_n^\varepsilon - \partial_t f_{n-1}, \frac{1}{\varepsilon} \{\mathbf{I} - \mathbf{P}_1\} \{f_n^\varepsilon - f_n\}) \\
&\leq \{ \varepsilon \|\partial_t f_n^\varepsilon\| + \|\nabla_x f_n^\varepsilon\|_\nu + \|\partial_t f_{n-1}\| \} \times \frac{1}{\varepsilon} \| \{\mathbf{I} - \mathbf{P}_1\} \{f_n^\varepsilon - f_n\} \|_\nu; \\
&(-\frac{1}{\sqrt{\mu}} \sum_{\substack{i+j=n \\ i,j \geq 1}} (E_i + v \times B_i) \cdot \nabla_v (\sqrt{\mu} g_j), \frac{1}{\varepsilon} \{\mathbf{I} - \mathbf{P}_1\} \{f_n^\varepsilon - f_n\}) \\
&\leq \{ \sum_{\substack{i+j=n \\ i,j \geq 1}} (\|E_i\| + \|B_i\|) (\|g_j\|_\nu + \|\nabla_v g_j\|) \} \times \frac{1}{\varepsilon} \| \{\mathbf{I} - \mathbf{P}_1\} \{f_n^\varepsilon - f_n\} \|_\nu.
\end{aligned}$$

For the remaining terms, we use Lemma 3.3 in [9] to get the upper bound as

$$\begin{aligned}
& \{ \{ (\sum_{i=1}^{n-1} \|g_i\|_\nu + \|g_n^\varepsilon\|_\nu) \cdot (\sum_{i=1}^{n-1} (\|E_i\| + \|B_i\|) + \|E_n^\varepsilon\| + \|B_n^\varepsilon\|) \} \\
&+ (\sum_{i=1}^{n-1} \|f_i\|_\nu + \|f_n^\varepsilon\|_\nu)^2 \} \times \frac{1}{\varepsilon} \| \{\mathbf{I} - \mathbf{P}_1\} \{f_n^\varepsilon - f_n\} \|_\nu
\end{aligned}$$

We therefore conclude that $\frac{1}{\varepsilon} \| \{\mathbf{I} - \mathbf{P}_1\} \{f_n^\varepsilon - f_n\} \|_\nu$ is uniformly bounded. Similarly, one can show that $\frac{1}{\varepsilon} \| \{\mathbf{I} - \mathbf{P}_2\} \{g_n^\varepsilon - g_n\} \|_\nu$ is also uniformly bounded. Hence there exist f_{n+1} , g_{n+1} such that

$$\{\mathbf{I} - \mathbf{P}_1\} \left\{ \frac{f_n^\varepsilon - f_n}{\varepsilon} \right\} \rightharpoonup f_{n+1}, \quad \{\mathbf{I} - \mathbf{P}_2\} \left\{ \frac{g_n^\varepsilon - g_n}{\varepsilon} \right\} \rightharpoonup g_{n+1} \quad \text{weakly in } \|\cdot\|_\nu.$$

Subtracting off Lf_{n+1} , $\mathcal{L}g_{n+1}$ on both sides of (3.6) and (3.7), we again isolate the zeroth order term. Letting $\varepsilon \rightarrow 0$ again, one can deduce (1.14) for $m = n - 1$.

We now turn to the derivation of various hydrodynamic equations based on (1.14) and (1.15). For the pure Boltzmann case, it is well known that the case $m = 1$ yields the celebrated nonlinear incompressible Navier-Stokes equations and $m > 1$, the Navier-Stokes-Fourier system of which derivation can be found in [9] (p.21-25). We take the same path to derive new dissipative PDE's from the rescaled Vlasov-Maxwell-Boltzmann system. We will use numerous results from [9]. First consider (1.14) when $m = -1$. It is equivalent to (1.21) and hence we get

$$(3.8) \quad f_1 = \{\rho_1 + v \cdot u_1 + (\frac{|v|^2 - 3}{2})\theta_1\}\sqrt{\mu} \quad \text{and} \quad g_1 = \sigma_1\sqrt{\mu}.$$

When $m = 0$, electromagnetic field equations become $\nabla \times B_1 = 0$, $\nabla \cdot B_1 = 0$, $\nabla \times E_1 = 0$, and $\nabla \cdot E_1 = \sigma_1$. Thus we may assume $B_1 = 0$ and $E_1 = \nabla\phi_1$ for some ϕ_1 which satisfies $\Delta\phi_1 = \sigma_1$ under the restriction $\int B_1 dx = 0$ and $\int E_1 dx = 0$; indeed, a nonzero constant B_1 does not produce any new macroscopic terms in (1.14) when $m = 1$, since

$$\begin{aligned} v \times B_1 \cdot \nabla_v(\sqrt{\mu}g_1) &= v \times B_1 \cdot \nabla_v(\sigma_1\mu) = v \times B_1 \cdot (-\sigma\mu)v = 0, \\ \langle \frac{1}{\sqrt{\mu}}v \times B_1 \cdot \nabla_v(\sqrt{\mu}f_1), \sqrt{\mu} \rangle &= \langle \frac{1}{\sqrt{\mu}}v \times B_1 \cdot \nabla_v(v \cdot u_1\mu), \sqrt{\mu} \rangle = 0, \end{aligned}$$

where the last equality is due to the integration by parts in v . Recalling the kernel of L and \mathcal{L} , by collision invariant property,

$$(3.9) \quad \langle v \cdot \nabla_x f_1, [1, v, |v|^2/2]\sqrt{\mu} \rangle = 0 \quad \text{and} \quad \langle v \cdot \nabla_x g_1 - E_1 \cdot v\sqrt{\mu}, \sqrt{\mu} \rangle = 0.$$

Plugging (3.8) into (3.9), the very first equation gives rise to the incompressibility (1.23) and Boussinesq relation (1.24). Microscopic part of f_2 and g_2 can be written as following by solving (1.14) when $m = 1$ for Lf_2 and $\mathcal{L}g_2$:

$$\begin{aligned} \{\mathbf{I} - \mathbf{P}_1\}f_2 &= L^{-1}\{-v \cdot \nabla_x f_1 + \Gamma(f_1, f_1)\}, \\ (3.10) \quad \{\mathbf{I} - \mathbf{P}_2\}g_2 &= \mathcal{L}^{-1}\{-v \cdot \nabla_x g_1 + \Gamma(g_1, f_1) + E_1 \cdot v\sqrt{\mu}\} \\ &= \mathcal{L}^{-1}(v\sqrt{\mu}) \cdot (-\nabla_x \sigma_1 + E_1) + \sigma_1 f_1, \end{aligned}$$

because of the following identity obtained from (1.13) and (1.21):

$$(3.11) \quad \mathcal{L}^{-1}\Gamma(g_1, f_1) = \sigma_1 \mathcal{L}^{-1}\Gamma(\sqrt{\mu}, f_1) = \sigma_1 \mathcal{L}^{-1}[-Lf_1 + \mathcal{L}f_1] = \sigma_1 f_1.$$

Notice that $\{\mathbf{I} - \mathbf{P}_1\}f_2$ and $\{\mathbf{I} - \mathbf{P}_2\}g_2$ are completely determined by already known functions. By collision invariant property, we get for $m = 1$

$$\begin{aligned} (3.12) \quad \langle \partial_t f_1 + v \cdot \nabla_x f_2 + \frac{1}{\sqrt{\mu}}E_1 \cdot \nabla_v(\sqrt{\mu}g_1), [1, v, |v|^2/2]\sqrt{\mu} \rangle &= 0, \\ \langle \partial_t g_1 + v \cdot \nabla_x g_2 + \frac{1}{\sqrt{\mu}}E_1 \cdot \nabla_v(\sqrt{\mu}f_1) - E_2 \cdot v\sqrt{\mu}, \sqrt{\mu} \rangle &= 0. \end{aligned}$$

The process is similar to the derivation from pure Boltzmann to Navier-Stokes except that now we have new terms to deal with; for instance, see [1, 13] for the computation of $\langle \partial_t f_1 + v \cdot \nabla_x f_2, [1, v, |v|^2/2]\sqrt{\mu} \rangle$. Here we pay attention to those extras such as electric field related terms. Note that magnetic fields are not involved

at this stage. The first equation of (3.12) is equivalent to

$$\begin{aligned} \partial_t \rho_1 + \nabla \cdot u_2 + \left\langle \frac{1}{\sqrt{\mu}} E_1 \cdot \nabla_v (\sqrt{\mu} g_1), \sqrt{\mu} \right\rangle &= 0, \\ \partial_t u_1 + u_1 \cdot \nabla u_1 + \nabla p_1 - \eta \Delta u_1 + \left\langle \frac{1}{\sqrt{\mu}} E_1 \cdot \nabla_v (\sqrt{\mu} g_1), v \sqrt{\mu} \right\rangle &= 0, \\ \frac{3}{2} \partial_t \{\rho_1 + \theta_1\} + \frac{5}{2} u_1 \cdot \nabla \theta_1 - \frac{5}{2} \kappa \Delta \theta_1 + \frac{5}{2} \nabla \cdot u_2 + \left\langle \frac{1}{\sqrt{\mu}} E_1 \cdot \nabla_v (\sqrt{\mu} g_1), \frac{|v|^2}{2} \sqrt{\mu} \right\rangle &= 0. \end{aligned}$$

Electric field related terms can be taken care of by integrating by parts in v :

$$\begin{aligned} \left\langle \frac{1}{\sqrt{\mu}} E_1 \cdot \nabla_v (\sqrt{\mu} g_1), \sqrt{\mu} \right\rangle &= E_1 \cdot \int \nabla_v (\sigma_1 \mu) dv = 0, \\ \left\langle \frac{1}{\sqrt{\mu}} E_1 \cdot \nabla_v (\sqrt{\mu} g_1), v \sqrt{\mu} \right\rangle &= E_1 \cdot \int \nabla_v (\sigma_1 \mu) v dv = -E_1 \int \sigma_1 \mu dv = -E_1 \sigma_1, \\ \left\langle \frac{1}{\sqrt{\mu}} E_1 \cdot \nabla_v (\sqrt{\mu} g_1), \frac{|v|^2}{2} \sqrt{\mu} \right\rangle &= E_1 \cdot \int \nabla_v (\sigma_1 \mu) \frac{|v|^2}{2} dv = -E_1 \cdot \left(\sigma_1 \int v \sqrt{\mu} dv \right) = 0. \end{aligned}$$

Thus the first equation in (3.12) leads to (1.31) for $m = 2$ as well as (1.25) and (1.28). Before going any further, let us integrate (3.12) over \mathbf{T}^3 and then, based on the above computation, we get

$$\frac{d}{dt} \int_{\mathbf{T}^3} u_1(t, x) dx = \frac{d}{dt} \int_{\mathbf{T}^3} \theta_1(t, x) dx = \frac{d}{dt} \int_{\mathbf{T}^3} \sigma_1(t, x) dx = 0$$

which assures the validity of (2.1) for $m = 1$. For the LHS of the second equation in (3.12), since

$$\left\langle \frac{1}{\sqrt{\mu}} E_1 \cdot \nabla_v (\sqrt{\mu} f_1) - E_2 \cdot v \sqrt{\mu}, \sqrt{\mu} \right\rangle = 0 \text{ and } \langle v \cdot \nabla_x \mathbf{P}_2 g_2, \sqrt{\mu} \rangle = 0,$$

first it is reduced to

$$\begin{aligned} &\langle \partial_t g_1 + v \cdot \nabla_x g_2 + \frac{1}{\sqrt{\mu}} E_1 \cdot \nabla_v (\sqrt{\mu} f_1) - E_2 \cdot v \sqrt{\mu}, \sqrt{\mu} \rangle \\ &= \partial_t \sigma_1 + \langle v \cdot \nabla_x \{\mathbf{I} - \mathbf{P}_2\} g_2, \sqrt{\mu} \rangle. \end{aligned}$$

By (3.10), the second term can be written as,

$$\begin{aligned} \langle v \cdot \nabla_x \{\mathbf{I} - \mathbf{P}_2\} g_2, \sqrt{\mu} \rangle &= \langle v \cdot \nabla_x \{ \mathcal{L}^{-1}(v \sqrt{\mu}) \cdot (-\nabla_x \sigma_1 + E_1) + \sigma_1 f_1 \}, \sqrt{\mu} \rangle \\ (3.13) \quad &= -\alpha \Delta \sigma_1 + \alpha \sigma_1 + \nabla \cdot (\sigma_1 u_1) \\ &= -\alpha \Delta \sigma_1 + \alpha \sigma_1 + u_1 \cdot \nabla \sigma_1, \end{aligned}$$

where

$$\alpha = \int_{\mathbb{R}^3} \mathcal{L}^{-1}(v \sqrt{\mu}) \cdot v \sqrt{\mu} dv > 0.$$

Hence we obtain (1.26) and it completes the case $m = 1$.

Next we move onto higher order systems; consider (1.14) for $m \geq 2$. We shall use an induction on m . First by collision invariant property, we get the following:

$$(3.14) \quad \begin{aligned} & \langle \partial_t f_m + v \cdot \nabla_x f_{m+1} + \frac{1}{\sqrt{\mu}} \sum_{\substack{i+j=m+1 \\ i,j \geq 1}} (E_i + v \times B_i) \cdot \nabla_v (\sqrt{\mu} g_j), [1, v, \frac{|v|^2}{2}] \sqrt{\mu} \rangle = 0, \\ & \langle \partial_t g_m + v \cdot \nabla_x g_{m+1} + \frac{1}{\sqrt{\mu}} \sum_{\substack{i+j=m+1 \\ i,j \geq 1}} (E_i + v \times B_i) \cdot \nabla_v (\sqrt{\mu} f_j) - E_{m+1} \cdot v \sqrt{\mu}, \sqrt{\mu} \rangle \\ & \hspace{15em} = 0, \end{aligned}$$

where $\langle \partial_t f_m + v \cdot \nabla_x f_{m+1}, [1, v, |v|^2/2] \sqrt{\mu} \rangle$ was computed for pure Boltzmann case in [9] (p.21-25). Here we will compute electromagnetic field related terms and then combine them with the result in [9].

Notice that for each $i, j \geq 1$,

$$\langle \frac{1}{\sqrt{\mu}} (E_i + v \times B_i) \cdot \nabla_v (\sqrt{\mu} g_j), \sqrt{\mu} \rangle = 0,$$

since each integrand is a perfect derivative in v . Hence the very first equation of (3.14) yields the $(m+1)$ -th order incompressibility condition:

$$(3.15) \quad \partial_t \rho_m + \nabla \cdot u_{m+1} = 0.$$

To derive the velocity equation, we look at the next electromagnetic field term:

$$\begin{aligned} & \sum_{\substack{i+j=m+1 \\ i,j \geq 1}} \langle \frac{1}{\sqrt{\mu}} (E_i + v \times B_i) \cdot \nabla_v (\sqrt{\mu} g_j), v \sqrt{\mu} \rangle \\ &= - \sum_{\substack{i+j=m+1 \\ i,j \geq 1}} \int (E_i + v \times B_i) g_j \sqrt{\mu} dv \quad (\text{by the integration by parts}) \\ &= - \sum_{\substack{i+j=m+1 \\ i,j \geq 1}} \{ E_i \nabla \cdot E_j - (\partial_t E_{j-1} - \nabla \times B_j) \times B_i \} \quad (\text{by (1.15)}) \end{aligned}$$

So the second part of the first equation in (3.14) with (4.8) in [9] is equivalent to

$$\begin{aligned} (3.16) \quad & \partial_t u_m + \langle v \cdot \nabla_x L^{-1} \{ -\partial_t f_{m-1} - v \cdot \nabla_x f_m - \frac{1}{\sqrt{\mu}} \sum_{\substack{i+j=m \\ i,j \geq 1}} (E_i + v \times B_i) \cdot \nabla_v (\sqrt{\mu} g_j) \\ & + \sum_{i+j=m+1} \Gamma(f_i, f_j) \}, v \sqrt{\mu} \rangle - \sum_{\substack{i+j=m+1 \\ i,j \geq 1}} \{ E_i \nabla \cdot E_j - (\partial_t E_{j-1} - \nabla \times B_j) \times B_i \} \\ & = -\nabla_x (\rho_{m+1} + \theta_{m+1}), \end{aligned}$$

where we have applied (1.29) to solve for the microscopic part $(\mathbf{I} - \mathbf{P}_1) f_{m+1}$. Any term of which index is lower than m in the above makes a contribution to the remainder R_m^u . In particular, all other electromagnetic terms are to be included in the remainder except for $E_m \nabla \cdot E_1$ and $E_1 \nabla \cdot E_m$. For the estimate of the rest, we refer [9] (p.21-25). As splitting $u_m = P_0 u_m + (I - P_0) u_m$, one can readily keep track of each term in (1.33), (3.2) and (3.5). In addition, to obtain the m -th order

Boussinesq relation (1.32), we use the pressure p_{m-1} in (3.5) to solve for $\rho_m + \theta_m$. Notice that by taking divergence of the $(m-1)$ -th order Navier-Stokes equation (1.33), we have another expression for p_{m-1} as

$$p_{m-1} = \Delta^{-1} \nabla \cdot \{-u_1 \cdot \nabla (P_0 u_{m-1}) - P_0 u_{m-1} \cdot \nabla u_1 + E_1 \sigma_m + E_m \sigma_1 + R_{m-1}^u\}.$$

For the temperature equation, we need to compute the following electromagnetic term:

$$\begin{aligned} & \sum_{\substack{i+j=m+1 \\ i,j \geq 1}} \langle \frac{1}{\sqrt{\mu}} (E_i + v \times B_i) \cdot \nabla_v (\sqrt{\mu} g_j), \frac{|v|^2}{2} \sqrt{\mu} \rangle \\ &= - \sum_{\substack{i+j=m+1 \\ i,j \geq 1}} \int (E_i + v \times B_i) \cdot v g_j \sqrt{\mu} dv \quad (\text{by the integration by parts}) \\ &= - \sum_{\substack{i+j=m+1 \\ i,j \geq 1}} E_i \cdot \int v g_j \sqrt{\mu} dv \quad (\text{since } V \times W \cdot V = 0 \text{ for } V, W \text{ vectors in } \mathbf{R}^3) \\ &= \sum_{\substack{i+j=m+1 \\ i,j \geq 1}} E_i \cdot (\partial_t E_{j-1} - \nabla \times B_j) \quad (\text{by (1.15)}) \end{aligned}$$

Then the last part of the first equation in (3.14) by using (4.8) in [9] becomes

$$\begin{aligned} & (3.17) \\ & \frac{5}{2} \partial_t \theta_m + \langle v \cdot \nabla_x L^{-1} \{-\partial_t f_{m-1} - v \cdot \nabla_x f_m - \frac{1}{\sqrt{\mu}} \sum_{\substack{i+j=m \\ i,j \geq 1}} (E_i + v \times B_i) \cdot \nabla_v (\sqrt{\mu} g_j) \\ & + \sum_{i+j=m+1} \Gamma(f_i, f_j)\}, \frac{|v|^2}{2} \sqrt{\mu} \rangle + \sum_{\substack{i+j=m+1 \\ i,j \geq 1}} E_i \cdot (\partial_t E_{j-1} - \nabla \times B_j) \\ & = \partial_t \{\rho_m + \theta_m\}, \end{aligned}$$

where (3.15) has been used. One can easily deduce (1.35) as well as (3.4).

Now we come to the second equation in (3.14). It is equivalent to

$$(3.18) \quad \partial_t \sigma_m + \langle v \cdot \nabla_x (\mathbf{I} - \mathbf{P}_2) g_{m+1}, \sqrt{\mu} \rangle = 0,$$

because $\langle \frac{1}{\sqrt{\mu}} \sum_{\substack{i+j=m+1 \\ i,j \geq 1}} (E_i + v \times B_i) \cdot \nabla_v (\sqrt{\mu} f_j) - E_{m+1} \cdot v \sqrt{\mu}, \sqrt{\mu} \rangle = 0$ and $\langle v \cdot \nabla_x \mathbf{P}_2 g_{m+1}, \sqrt{\mu} \rangle = 0$. Solving for the microscopic part $(\mathbf{I} - \mathbf{P}_2) g_{m+1}$ by (1.30) and plugging it into (3.18), we deduce that

$$\begin{aligned} & (3.19) \\ & \partial_t \sigma_m + \langle v \cdot \nabla_x \mathcal{L}^{-1} \{-(\mathbf{I} - \mathbf{P}_2)(v \cdot \nabla_x \mathbf{P}_2 g_m) + E_m \cdot v \sqrt{\mu} + \Gamma(g_1, \mathbf{P}_1 f_m) \\ & \quad + \Gamma(\mathbf{P}_2 g_m, f_1)\}, \sqrt{\mu} \rangle \\ &= \langle v \cdot \nabla_x \mathcal{L}^{-1} \{\partial_t (\mathbf{I} - \mathbf{P}_2) g_{m-1} + (\mathbf{I} - \mathbf{P}_2)(v \cdot \nabla_x (\mathbf{I} - \mathbf{P}_2) g_m) - \Gamma(g_1, (\mathbf{I} - \mathbf{P}_1) f_m) \\ & \quad - \Gamma((\mathbf{I} - \mathbf{P}_2) g_m, f_1) + \frac{1}{\sqrt{\mu}} \sum_{\substack{i+j=m \\ i,j \geq 1}} (E_i + v \times B_i) \cdot \nabla_v (\sqrt{\mu} f_j) \\ & \quad + \sum_{\substack{i+j=m+1 \\ i,j \geq 2}} \Gamma(g_i, f_j)\}, v \sqrt{\mu} \rangle. \end{aligned}$$

Note that each term in the RHS of (3.19) essentially has an index of lower than m and hence is part of (3.3). Terms in the LHS can be computed in the same way as in (3.13). Recalling $\mathbf{P}_2 g_m = \sigma_m \sqrt{\mu}$, we get

$$\begin{aligned} \langle v \cdot \nabla_x \mathcal{L}^{-1} \{ (\mathbf{I} - \mathbf{P}_2)(v \cdot \nabla_x \mathbf{P}_2 g_m) \}, \sqrt{\mu} \rangle &= \alpha \Delta \sigma_m, \\ \langle v \cdot \nabla_x \mathcal{L}^{-1} \{ E_m \cdot v \sqrt{\mu} \}, \sqrt{\mu} \rangle &= \alpha \sigma_m, \\ \langle v \cdot \nabla_x \mathcal{L}^{-1} \{ \Gamma(g_1, \mathbf{P}_1 f_m) + \Gamma(\mathbf{P}_2 g_m, f_1) \}, \sqrt{\mu} \rangle &= \nabla \cdot (\sigma_1 u_m) + \nabla \cdot (\sigma_m u_1), \end{aligned}$$

where we have used the fact $\Gamma(\sqrt{\mu}, \mathbf{P}_1 f_m) = \mathcal{L} \mathbf{P}_1 f_m$ and $\Gamma(\sqrt{\mu}, f_1) = \mathcal{L} f_1$ which follow from the following observation: for any h with $Lh = 0$,

$$\mathcal{L}h = -\Gamma(h, \sqrt{\mu}) = \Gamma(\sqrt{\mu}, h), \quad \text{since } -Lh = \Gamma(h, \sqrt{\mu}) + \Gamma(\sqrt{\mu}, h).$$

Therefore, (1.34) can be derived with the remainder R_m^σ (3.3).

As for field equations, (1.15) with the definition of $\mathbf{P}_2 g_m$ immediately leads to (1.36) and (1.37). To get the compatibility conditions (1.38), take the integral in x of field equations in (1.15) and then the periodic boundary conditions give rise to

$$\frac{d}{dt} \int_{\mathbb{T}^3} B_m dx = 0, \quad \frac{d}{dt} \int_{\mathbb{T}^3} E_m dx = - \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} v g_{m+1} \sqrt{\mu} dv dx.$$

Thus $\int_{\mathbb{T}^3} B_m dx = 0$ may be assumed for all time. The dynamics of $\int_{\mathbb{T}^3} E_m dx$ is more complicated. First note that $\int \int \mathbf{P}_2 g_m v \sqrt{\mu} dv dx = 0$. By using (1.30), we have

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{T}^3} E_m dx &= - \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} (\mathbf{I} - \mathbf{P}_2) g_{m+1} v \sqrt{\mu} dv dx \\ &= - \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} v \sqrt{\mu} \mathcal{L}^{-1} \{ -\partial_t g_{m-1} - v \cdot \nabla_x g_m + \sum_{\substack{i+j=m+1 \\ i,j \geq 1}} \Gamma(g_i, f_j) \\ &\quad - \frac{1}{\sqrt{\mu}} \sum_{\substack{i+j=m \\ i,j \geq 1}} (E_i + v \times B_i) \cdot \nabla_v (\sqrt{\mu} f_j) + E_m \cdot v \sqrt{\mu} \} dv dx \\ &= \alpha \int_{\mathbb{T}^3} E_m dx + \ell_{m-1}, \end{aligned}$$

where

$$\begin{aligned} \ell_{m-1} &\equiv \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \partial_t g_{m-1} \mathcal{L}^{-1}(v \sqrt{\mu}) dv dx \\ (3.20) \quad &+ \sum_{\substack{i+j=m \\ i,j \geq 1}} \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \frac{1}{\sqrt{\mu}} (E_i + v \times B_i) \cdot \nabla_v (\sqrt{\mu} f_j) \mathcal{L}^{-1}(v \sqrt{\mu}) dv dx. \end{aligned}$$

Note that ℓ_{m-1} only contains terms of index lower than m and $\int E_m dx$ at $t = 0$ for $m \geq 2$ can be arbitrarily given. Other terms have vanished after the integration because of either a perfect derivative or collision invariant property.

It remains to verify conservation laws (2.1) for $m \geq 2$ to finish Lemma. Firstly, it is easy to see

$$\frac{d}{dt} \int_{\mathbb{T}^3} \rho_m dx = 0 \quad \text{and} \quad \frac{d}{dt} \int_{\mathbb{T}^3} \sigma_m dx = 0,$$

by integrating (3.15) and (3.18) over \mathbb{T}^3 . We give the detailed computation for the average of the velocity and the similar argument can be applied to the temperature

field. Integrate (3.16) to get

$$(3.21) \quad \frac{d}{dt} \int_{\mathbb{T}^3} u_m dx = \sum_{\substack{i+j=m+1 \\ i,j \geq 1}} \int_{\mathbb{T}^3} \{E_i \nabla \cdot E_j - (\partial_t E_{j-1} - \nabla \times B_j) \times B_i\} dx.$$

To simplify the RHS, we observe the following:

$$\begin{aligned} \int_{\mathbb{T}^3} E_i (\nabla \cdot E_j) + E_j (\nabla \cdot E_i) dx &= - \int_{\mathbb{T}^3} (\nabla \times E_i) \times E_j + (\nabla \times E_j) \times E_i dx \\ &= \int_{\mathbb{T}^3} \partial_t B_{i-1} \times E_j + \partial_t B_{j-1} \times E_i dx, \\ \int_{\mathbb{T}^3} (\nabla \times B_j) \times B_i + (\nabla \times B_i) \times B_j dx &= 0 \quad (\text{since } \nabla \cdot B_i = \nabla \cdot B_j = 0), \end{aligned}$$

where we have integrated by parts. Thus the RHS of (3.21) is equivalent to

$$\sum_{\substack{i+j=m+1 \\ i,j \geq 1}} \int_{\mathbb{T}^3} \partial_t B_{i-1} \times E_j - \partial_t E_{j-1} \times B_i dx$$

and in turn by rearranging indices we deduce

$$\frac{d}{dt} \int_{\mathbb{T}^3} u_m dx = - \frac{d}{dt} \sum_{\substack{i+j=m \\ i,j \geq 1}} \int_{\mathbb{T}^3} E_i \times B_j dx.$$

Moreover, $\int_{\mathbb{T}^3} u_m dx = \int_{\mathbb{T}^3} P_0 u_m dx$, since $\{I - P_0\}u_m = \nabla q_m$ for some scalar function q_m by the Helmholtz-Hodge decomposition. Therefore, the desired result follows. \square

4. THE DIFFUSIVE COEFFICIENTS

We now show that we can determine the coefficients $f_1, f_2, \dots, f_m; g_1, g_2, \dots, g_m$ and $E_1, \dots, E_m; B_1, \dots, B_m$ by giving the initial divergent free part of velocity field $u_i^0(x)$, the temperature field $\theta_i^0(x)$, the concentration difference field $\sigma_i^0(x)$ and the average value e_i of initial electric field E_i^0 with $e_1 = 0$.

Proof. (of Theorem 2.1:) We use the induction over r . First consider the case $r = 1$. We need to solve the system (1.25)-(1.28) with (1.23) and $\int u_1 = \int \theta_1 = \int \sigma_1 = 0$. Once a priori estimate is given, the existence and the uniqueness of a solution $[u_1(t, x), \theta_1(t, x), \sigma_1(t, x), \phi_1(t, x)]$ follow from the standard iteration argument via the fixed point theorem. Thus it suffices to illustrate a priori energy estimate in the subsequence. First we take ∂_γ derivatives of (1.25)-(1.28):

$$(4.1) \quad \begin{aligned} \partial_t \partial_\gamma u_1 + u_1 \nabla \partial_\gamma u_1 + \nabla \partial_\gamma p_1 - \eta \Delta \partial_\gamma u_1 &= -\partial_{\gamma_1} u_1 \nabla \partial_{\gamma_2} u_1 + \partial_\gamma (\sigma_1 \nabla \phi_1), \\ \partial_t \partial_\gamma \sigma_1 + u_1 \nabla \partial_\gamma \sigma_1 - \alpha \Delta \partial_\gamma \sigma_1 + \alpha \partial_\gamma \sigma_1 &= -\partial_{\gamma_1} u_1 \nabla \partial_{\gamma_2} \sigma_1, \\ \partial_t \partial_\gamma \theta_1 + u_1 \nabla \partial_\gamma \theta_1 - \kappa \Delta \partial_\gamma \theta_1 &= -\partial_{\gamma_1} u_1 \nabla \partial_{\gamma_2} \theta_1, \\ \Delta \partial_\gamma \phi_1 &= \partial_\gamma \sigma_1, \quad \nabla \cdot u_1 = 0. \end{aligned}$$

The last three summations are over $\gamma_1 + \gamma_2 = \gamma$ with $|\gamma_1| \geq 1$. Let $\gamma = 0$. Multiply first three equation in (4.1) by $[u_1, \sigma_1, \theta_1]$, integrate over \mathbb{T}^3 and by the incompressibility we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left[\int |u_1|^2 + |\sigma_1|^2 + |\theta_1|^2 dx \right] + \eta \int |\nabla u_1|^2 dx + \alpha \int |\nabla \sigma_1|^2 dx + \kappa \int |\nabla \theta_1|^2 dx \\ & + \alpha \int |\sigma_1|^2 dx = \int \sigma_1 \nabla \phi_1 \cdot u_1 dx. \end{aligned}$$

Note that the RHS can be absorbed into the LHS assuming M is sufficiently small at time t , since

$$\int \sigma_1 \nabla \phi_1 \cdot u_1 dx \leq (\sup_x |u_1|) \|\sigma_1\| \cdot \|\nabla \phi_1\| \leq CM \|\sigma_1\|^2,$$

where we have used the Sobolev embedding theorem and the L^2 estimate for the poisson equation $\Delta \phi_1 = \sigma_1$. Similarly, the energy estimate for $|\gamma| \leq 2$ shows

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left[\int |\partial_\gamma u_1|^2 + |\partial_\gamma \sigma_1|^2 + |\partial_\gamma \theta_1|^2 dx \right] + \eta \int |\nabla \partial_\gamma u_1|^2 dx + \alpha \int |\nabla \partial_\gamma \sigma_1|^2 dx \\ & + \kappa \int |\nabla \partial_\gamma \theta_1|^2 dx + \alpha \int |\partial_\gamma \sigma_1|^2 dx \\ & \leq \sup_x (|u_1| + |\theta_1| + |\sigma_1|) (\|\partial_\gamma u_1\|^2 + \|\nabla \partial_\gamma u_1\|^2 + \|\partial_\gamma \sigma_1\|^2 + \|\nabla \partial_\gamma \sigma_1\|^2 + \|\partial_\gamma \theta_1\|^2 \\ & + \|\nabla \partial_\gamma \theta_1\|^2 + \|\nabla \partial_\gamma \phi_1\|^2). \end{aligned}$$

Each term in the RHS can be absorbed into the LHS such that

$$\frac{1}{2} \frac{d}{dt} (\|u_1\|_{H^2}^2 + \|\sigma_1\|_{H^2}^2 + \|\theta_1\|_{H^2}^2) + \frac{\eta}{2} \|\nabla u_1\|_{H^2}^2 + \frac{\alpha}{2} \|\sigma_1\|_{H^3}^2 + \frac{\kappa}{2} \|\nabla \theta_1\|_{H^2}^2 \leq 0.$$

Applying the Poincaré inequality, for $\lambda = \frac{1}{4} \min\{\eta, \alpha, \kappa\} > 0$ we have

$$\|u_1(t)\|_{H^2} + \|\sigma_1(t)\|_{H^2} + \|\theta_1(t)\|_{H^2} \leq C e^{-\lambda t} \{\|u_1^0\|_{H^2} + \|\sigma_1^0\|_{H^2} + \|\theta_1^0\|_{H^2}\},$$

which immediately implies the existence, uniqueness and exponential decay of $u_1(t, x)$, $\sigma_1(t, x)$ and $\theta_1(t, x)$.

Next we turn to high order derivative cases. We claim that for $s \geq 3$, there exists a polynomial $U_s \geq 0$ with $U_s(0) = 0$ such that

$$(4.2) \quad \|u_1(t)\|_{H^s} + \|\theta_1(t)\|_{H^s} + \|\sigma_1(t)\|_{H^s} \leq e^{-\lambda t} U_s (\|u_1^0\|_{H^s} + \|\theta_1^0\|_{H^s} + \|\sigma_1^0\|_{H^s}).$$

Separating the case of $|\gamma_1| = 1$ or $|\gamma_1| = s$, and the case of $|\gamma_1| \leq s-1$, $|\gamma_2| \leq s-2$, we estimate the L^2 norm of the RHS' in (4.1) by

$$\begin{aligned} & \{C(\|u_1\|_{H^2} + \|\sigma_1\|_{H^2}) + \xi\} \left\{ \sum_{|\gamma|=s+1} (\|\partial_\gamma u_1\| + \|\partial_\gamma \sigma_1\| + \|\partial_\gamma \theta_1\|) + \sum_{|\gamma|=s} \|\partial_\gamma \nabla \phi_1\| \right\} \\ & + C_\xi \{\|u_1\|_{H^{s-1}} + \|\sigma_1\|_{H^{s-1}} + \|\theta_1\|_{H^{s-1}} + \|\nabla \phi_1\|_{H^{s-1}}\}^2, \end{aligned}$$

for any small number $\xi > 0$, by an interpolation in the Sobolev Space. We then use standard energy estimate for $\partial_\gamma u_1$, $\partial_\gamma \sigma_1$ and $\partial_\gamma \theta_1$ to get ($\nabla \cdot u_1 = 0$)

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \sum_{|\gamma|=s} \{ \|\partial_\gamma u_1\|^2 + \|\partial_\gamma \sigma_1\|^2 + \|\partial_\gamma \theta_1\|^2 \} \\ & + \sum_{|\gamma|=s} \{ \eta \|\nabla \partial_\gamma u_1\|^2 + \kappa \|\nabla \partial_\gamma \theta_1\|^2 + \alpha \|\partial_\gamma \sigma_1\|^2 + \alpha \|\nabla \partial_\gamma \sigma_1\|^2 \} \leq \\ & \{ C(\|u_1\|_{H^2} + \|\sigma_1\|_{H^2}) + \xi \} \{ \sum_{|\gamma|=s+1} \|\partial_\gamma u_1\| + \|\partial_\gamma \sigma_1\| + \|\partial_\gamma \theta_1\| + \sum_{|\gamma|=s} \|\nabla \partial_\gamma \phi_1\| \}^2 \\ & + C_\xi \{ \|u_1\|_{H^{s-1}} + \|\sigma_1\|_{H^{s-1}} + \|\theta_1\|_{H^{s-1}} + \|\nabla \phi_1\|_{H^{s-1}} \}^4. \end{aligned}$$

Note that $\|\nabla \phi_1\|_{H^{s-1}} \leq C \|\sigma_1\|_{H^{s-1}}$. For $C(\|u_1\|_{H^2} + \|\sigma_1\|_{H^2})$ and ξ sufficiently small, we deduce from the Poincaré inequality and an induction for $s-1$ over the last lower order term that

$$\begin{aligned} & \sum_{|\gamma| \leq s} \|\partial_\gamma u_1(t)\| + \|\partial_\gamma \sigma_1(t)\| + \|\partial_\gamma \theta_1(t)\| \\ & \leq e^{-\lambda t} \sum_{|\gamma| \leq s} \|\partial_\gamma u_1(0)\| + \|\partial_\gamma \sigma_1(0)\| + \|\partial_\gamma \theta_1(0)\| \\ & + \int_0^t e^{-\lambda\{t-\tau\}} e^{-2\lambda\tau} \{ U_{s-1}(\|u_1^0\|_{H^{s-1}} + \|\sigma_1^0\|_{H^{s-1}} + \|\theta_1^0\|_{H^{s-1}}) \}^2 d\tau \\ & \leq e^{-\lambda t} U_s(\|u_1(0)\|_{H^s} + \|\sigma_1(0)\|_{H^s} + \|\theta_1(0)\|_{H^s}). \end{aligned}$$

We thus conclude our claim (4.2).

Now let us turn to general space-time derivatives ∂_τ . Notice that

$$\Delta p_1 = \nabla \cdot \{ \sigma_1 \nabla \phi_1 - u_1 \cdot \nabla u_1 \}.$$

We then use repeatedly the equations (1.25), (1.26) and (1.28) to solve for temporal derivatives to get

$$\begin{aligned} \sum_{|\tau| \leq s} \{ \|\partial_\tau u_1(t)\| + \|\partial_\tau \sigma_1(t)\| + \|\partial_\tau \theta_1(t)\| \} & \leq e^{-\lambda t} U_{2s}(\|u_1(0)\|_{H^{2s}} + \|\sigma_1(0)\|_{H^{2s}} \\ & + \|\theta_1(0)\|_{H^{2s}}). \end{aligned}$$

Notice that we need twice many spatial derivatives now for the initial data. Finally, since

$$f_1 \equiv v \cdot u_1 \sqrt{\mu} + \left\{ \frac{|v|^2}{2} - \frac{5}{2} \right\} \theta_1 \sqrt{\mu}, \quad \text{and} \quad g_1 \equiv \sigma_1 \sqrt{\mu},$$

it follows that for any $s \geq 0$ and β ,

$$\begin{aligned} \sum_{|\tau| \leq s} \| [\partial_\tau^\beta f_1(t), \partial_\tau^\beta g_1(t)] \|_\nu & \leq C \sum_{|\tau| \leq s} \{ \|\partial_\tau u_1(t)\| + \|\partial_\tau \sigma_1(t)\| + \|\partial_\tau \theta_1(t)\| \} \\ & \leq e^{-\lambda t} U_{2s}(\|u_1^0\|_{H^{2s}} + \|\sigma_1^0\|_{H^{2s}} + \|\theta_1^0\|_{H^{2s}}). \end{aligned}$$

Recall that $E_1 = \nabla \phi_1$ with $\Delta \phi_1 = \sigma_1$ and $B_1 = 0$. Our theorem thus is valid for $r = 1$.

Assume that $f_1, f_2, \dots, f_r, g_1, g_2, \dots, g_r, E_1, E_2, \dots, E_r$, and $B_1 (= 0), B_2, \dots, B_r$ have been constructed to satisfy (1.29)-(1.38) for up to $r \geq 1$. We now construct f_{r+1} ,

g_{r+1} in two steps.

Step One : Construct the microscopic part $\{\mathbf{I} - \mathbf{P}_1\}f_{r+1}$ and $\{\mathbf{I} - \mathbf{P}_2\}g_{r+1}$ from the microscopic equation (1.14).

In order to solve for $\{\mathbf{I} - \mathbf{P}_1\}f_{r+1}$ and $\{\mathbf{I} - \mathbf{P}_2\}g_{r+1}$, we need (3.14) for $m = r - 1$, which are equivalent to (3.15), (3.16), (3.17) and (3.19) for $m = r - 1$, to be true. By the induction hypothesis, the r -th incompressibility condition (1.31) and the $(r - 1)$ -th temperature equation (1.35) imply that (3.15) and (3.17) hold for $m = r - 1$. And $(r - 1)$ -th concentration difference equation (1.34) leads to (3.19) for $m = r - 1$. Finally, applying the r -th order Boussinesq relation (1.32) into the $(r - 1)$ -th order Vlasov-Navier-Stokes system (1.33), we conclude that the pressure p_{r-1} is given by (3.5) for $m = r - 1$, which also implies that (3.16) is valid for $r - 1$. Therefore, we can solve $\{\mathbf{I} - \mathbf{P}_1\}f_{r+1}$ and $\{\mathbf{I} - \mathbf{P}_2\}g_{r+1}$ from (1.14) such that

$$\begin{aligned} L\{\mathbf{I} - \mathbf{P}_1\}f_{r+1} &= -\partial_t f_{r-1} - v \cdot \nabla_x f_r - \frac{1}{\sqrt{\mu}} \sum_{\substack{i+j=r \\ i,j \geq 1}} (E_i + v \times B_i) \cdot \nabla_v (\sqrt{\mu} g_j) \\ &\quad + \sum_{\substack{i+j=r+1 \\ i,j \geq 1}} \Gamma(f_i, f_j); \\ \mathcal{L}\{\mathbf{I} - \mathbf{P}_2\}g_{r+1} &= -\partial_t g_{r-1} - v \cdot \nabla_x g_r - \frac{1}{\sqrt{\mu}} \sum_{\substack{i+j=r \\ i,j \geq 1}} (E_i + v \times B_i) \cdot \nabla_v (\sqrt{\mu} f_j) \\ &\quad + E_r \cdot v \sqrt{\mu} + \sum_{\substack{i+j=r+1 \\ i,j \geq 1}} \Gamma(g_i, f_j). \end{aligned}$$

We next estimate such microscopic parts. Since two expressions in the above have almost same structure, we compute only one term $\{\mathbf{I} - \mathbf{P}_2\}g_{r+1}$. The same argument can be applied to $\{\mathbf{I} - \mathbf{P}_1\}f_{r+1}$. Taking ∂_τ^β derivatives, we get

$$\begin{aligned} &(\partial_\tau^\beta \mathcal{L}\{\mathbf{I} - \mathbf{P}_2\}g_{r+1}, \partial_\tau^\beta \{\mathbf{I} - \mathbf{P}_2\}g_{r+1}) \\ &= (-\partial_\tau^\beta \partial_t g_{r-1} - \partial_\tau^\beta \{v \cdot \nabla_x g_r\} - \partial_\tau^\beta \left\{ \frac{1}{\sqrt{\mu}} \sum_{\substack{i+j=r \\ i,j \geq 1}} (E_i + v \times B_i) \cdot \nabla_v (\sqrt{\mu} f_j) \right\} \\ &\quad + \partial_\tau^\beta \{E_r \cdot v \sqrt{\mu}\} + \partial_\tau^\beta \sum_{\substack{i+j=r+1 \\ i,j \geq 1}} \Gamma(g_i, f_j), \partial_\tau^\beta \{\mathbf{I} - \mathbf{P}_2\}g_{r+1}). \end{aligned}$$

Applying Lemma 3.3 in [9], we first get ($C_\beta = 0$ if $\beta = 0$)

$$\begin{aligned} &\frac{1}{2} \|\partial_\tau^\beta \{\mathbf{I} - \mathbf{P}_2\}g_{r+1}\|_\nu^2 - C_\beta \|\{\mathbf{I} - \mathbf{P}_2\}g_{r+1}\|_\nu^2 \\ &\leq (-\partial_t \partial_\tau^\beta g_{r-1} - \partial_\tau^\beta \{v \cdot \nabla_x g_r\} + \partial_\tau^\beta \{E_r \cdot v \sqrt{\mu}\}, \partial_\tau^\beta \{\mathbf{I} - \mathbf{P}_2\}g_{r+1}) \\ &\quad + (\partial_\tau^\beta \sum_{\substack{i+j=r+1 \\ i,j \geq 1}} \Gamma(g_i, f_j), \partial_\tau^\beta \{\mathbf{I} - \mathbf{P}_2\}g_{r+1}) \\ &\quad + (-\partial_\tau^\beta \left\{ \frac{1}{\sqrt{\mu}} \sum_{\substack{i+j=r \\ i,j \geq 1}} (E_i + v \times B_i) \cdot \nabla_v \sqrt{\mu} f_j \right\}, \partial_\tau^\beta \{\mathbf{I} - \mathbf{P}_2\}g_{r+1}) \\ &\equiv (I) + (II) + (III). \end{aligned} \tag{4.3}$$

Since $|\partial^\beta \{v\sqrt{\mu}\}|$ is uniformly bounded, by using (1.40), (I) is bounded by

$$\{||\partial_t \partial_\tau^\beta g_{r-1}|| + ||\partial_\tau^\beta \nabla_x g_r||_\nu + ||\partial_\tau^{\beta-\beta_1} \nabla_x g_r|| + ||\partial_\tau E_r||\} \times ||\partial_\tau^\beta \{\mathbf{I} - \mathbf{P}_2\} g_{r+1}||_\nu,$$

where $|\beta_1| = 1$. Applying Lemma 3.3 in [9] again, the nonlinear collision term (II) is bounded by

$$\sum_{\substack{i+j=r+1 \\ i,j \geq 1}} \{||\partial_\tau^\beta g_i|| \cdot ||\partial_\tau^\beta f_j||_\nu + ||\partial_\tau^\beta g_i||_\nu \cdot ||\partial_\tau^\beta f_j||\} \times ||\partial_\tau^\beta \{\mathbf{I} - \mathbf{P}_2\} g_{r+1}||_\nu.$$

To control the electromagnetic terms (III), we introduce the following inequality.

Claim. There exists $C > 0$ such that for $|\tau| \leq s$ and $\beta_1 \leq \beta$,

$$\begin{aligned} & (\partial_\tau^\beta \{ \frac{1}{\sqrt{\mu}} (E + v \times B) \cdot \nabla_v (\sqrt{\mu} f) \}, \partial_\tau^\beta g) \\ (4.4) \quad & \leq C \sum_{|\tilde{\tau}| \leq \max\{s, 3\}} (||\partial_{\tilde{\tau}} E|| \cdot ||\partial_{\tilde{\tau}}^\beta \nabla_v f|| + ||\partial_{\tilde{\tau}} B|| \cdot ||\partial_{\tilde{\tau}}^{\beta_1} \nabla_v f||_\nu \\ & \quad + ||\partial_{\tilde{\tau}} E|| \cdot ||\partial_{\tilde{\tau}}^{\beta_1} f||) \cdot ||\partial_\tau^\beta g||_\nu. \end{aligned}$$

The direct result of the claim is that (III) is bounded by

$$\sum_{\substack{i+j=r \\ i,j \geq 1}} \sum_{|\tilde{\tau}|} \{||[\partial_{\tilde{\tau}} E_i, \partial_{\tilde{\tau}} B_i]|| \cdot ||\partial_{\tilde{\tau}}^\beta \nabla_v f_j||_\nu + ||\partial_{\tilde{\tau}} E_i|| \cdot ||\partial_{\tilde{\tau}}^\beta f_j||\} \times ||\partial_\tau^\beta \{\mathbf{I} - \mathbf{P}_2\} g_{r+1}||_\nu,$$

where $|\tilde{\tau}| \leq \max\{|\tau|, 3\}$. From the induction hypothesis (2.2) for $f_1, \dots, f_r; g_1, \dots, g_r$ and $E_1, \dots, E_r; B_1, \dots, B_r$, consequently, we deduce that for $|\tau| \leq s$, (there are $s+1$ derivatives for f_r, g_r)

$$\begin{aligned} & ||[\partial_\tau^\beta \{\mathbf{I} - \mathbf{P}_1\} f_{r+1}, \partial_\tau^\beta \{\mathbf{I} - \mathbf{P}_2\} g_{r+1}]||_\nu \\ (4.5) \quad & \leq C e^{-\lambda t} U_{r+1} \left(\sum_{i=1}^r \{||u_i^0||_{H^{2s+2+4(r-i)}} + ||\theta_i^0||_{H^{2s+2+4(r-i)}} + ||\sigma_i^0||_{H^{2s+2+4(r-i)}}\} \right). \end{aligned}$$

To complete (4.5), it remains to prove the above claim.

Proof of Claim: First we note that

$$\frac{1}{\sqrt{\mu}} (E + v \times B) \cdot \nabla_v (\sqrt{\mu} f) = (E + v \times B) \cdot \nabla_v f - E \cdot \frac{v\sqrt{\mu}}{2} f.$$

We compute only one term and other terms can be treated similarly.

$$\begin{aligned} (\partial_\tau^\beta \{E \cdot \nabla_v f\}, \partial_\tau^\beta g) &= \int \int \partial_{\tau_1} E \cdot (\partial_{\tau_2}^\beta \nabla_v f) \partial_\tau^\beta g dv dx \quad (\text{where } \tau_1 + \tau_2 = \tau) \\ &\leq \left(\int \int |\partial_{\tau_1} E \cdot (\partial_{\tau_2}^\beta \nabla_v f)|^2 dv dx \right)^{\frac{1}{2}} ||\partial_\tau^\beta g|| \\ &\leq C \left(\sum_{|\tilde{\tau}| \leq \max\{s, 3\}} ||\partial_{\tilde{\tau}} E|| \cdot ||\partial_{\tilde{\tau}}^\beta \nabla_v f|| \right) \cdot ||\partial_\tau^\beta g|| \end{aligned}$$

At the last step we have applied the Sobolev imbedding theorem. Note that $\min\{|\tau_1|, |\tau_2|\} + 2 \leq \max\{|\tau|, 3\}$. Therefore, this finishes the first step.

Step Two : Construct the hydrodynamic field $\mathbf{P}_1 f_{r+1}$ and $\mathbf{P}_2 g_{r+1}$ i.e. ρ_{r+1} , $u_{r+1}, \theta_{r+1}, \sigma_{r+1}$ and the electromagnetic field E_{r+1} and B_{r+1} .

First of all, we recall that $\{I - P_0\}u_{r+1} = \nabla q_{r+1}$ for some scalar function q_{r+1} and hence from (1.31) one obtains

$$\Delta q_{r+1} = -\partial_t \rho_r,$$

where $\{I - P_0\}u_{r+1}$ has average zero. Hence the elliptic L^2 estimates lead that $\sum_{|\tau| \leq s} \|\partial_\tau \{I - P_0\}u_{r+1}\|$ is bounded by $\sum_{|\tau| \leq s} \|\partial_\tau \partial_t \rho_r\|$, which is bounded by the RHS of (4.5) by the induction hypothesis.

We determine $\rho_{r+1} + \theta_{r+1}$ from the Boussinesq relation (1.32):

$$\begin{aligned} \rho_{r+1} + \theta_{r+1} &= \Delta^{-1} \nabla \cdot P_0 \{-u_1 \cdot \nabla \{P_0 u_r\} - u_r \cdot \nabla \{P_0 u_1\} + R_r^u\} \\ &\quad + \left\langle \frac{|v|^2 \sqrt{\mu}}{3}, L^{-1} \{v \cdot \nabla_x f_{r-1} + \Gamma(f_1, f_r)\} \right\rangle - \frac{5}{2} \theta_1 \theta_r - u_r \cdot u_1. \end{aligned}$$

Notice that by the formula (3.2) and the induction hypothesis, we easily conclude $\|\Delta^{-1} \nabla \cdot P_0 \partial_\tau R_r^u\|$ is bounded by the RHS of (4.5). Hence $\sum_{|\tau| \leq s} \|\partial_\tau \{\rho_{r+1} + \theta_{r+1}\}\|$ is bounded again by the RHS of (4.5).

Finally, to determine the remaining $P_0 u_{r+1}, \sigma_{r+1}, \theta_{r+1}$ and E_{r+1}, B_{r+1} we solve the $(r+1)$ -th order *linear* Vlasov-Navier-Stokes-Fourier system (1.33)-(1.38). By the standard energy method ($\nabla \cdot P_0 u_{r+1} = 0$), there is the unique solution $[P_0 u_{r+1}, \sigma_{r+1}, \theta_{r+1}, E_{r+1}, B_{r+1}]$. We illustrate the energy estimates in the subsequence. First, B_{r+1} is estimated and other terms will be done all together. From (1.37) and the Poincaré inequality, we get

$$\|\partial_\tau \nabla B_{r+1}\| \leq \| \{\mathbf{I} - \mathbf{P}_2\} \partial_\tau g_{r+1} \| + \|\partial_\tau \partial_t E_r\|, \quad \|B_{r+1}\| \leq C \|\nabla B_{r+1}\|.$$

By (4.5) and the induction hypothesis, $\sum_{|\tau| \leq s} \|\partial_\tau B_{r+1}\|$ is bounded by the desired quantity.

Next we move onto other terms. Take ∂_τ derivatives of (1.33) when $m = r+1$, multiply by $\partial_\tau P_0 u_{r+1}$ and integrate over \mathbb{T}^3 to get

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\partial_\tau P_0 u_{r+1}\|^2 + \eta \|\nabla \partial_\tau P_0 u_{r+1}\|^2 \\ &= -(\partial_\tau \{u_1 \cdot \nabla P_0 u_{r+1} + P_0 u_{r+1} \cdot \nabla u_1\}, \partial_\tau P_0 u_{r+1}) \\ &\quad + (\partial_\tau \{E_1 \sigma_{r+1} + E_{r+1} \sigma_1\}, \partial_\tau P_0 u_{r+1}) + (\partial_\tau R_{r+1}^u, \partial_\tau P_0 u_{r+1}). \end{aligned}$$

By the Cauchy-Schwartz inequality and Sobolev imbedding theorem, first two terms in the RHS can be estimated as follows:

$$\begin{aligned} &(\partial_\tau \{u_1 \cdot \nabla P_0 u_{r+1} + P_0 u_{r+1} \cdot \nabla u_1\}, \partial_\tau P_0 u_{r+1}) \\ &\leq \frac{\eta}{8} \sum_{|\tau| \leq s+1} \|\partial_\tau P_0 u_{r+1}\|^2 + C_\eta \sum_{|\tau| \leq s+2} \|\partial_\tau u_1\|^4; \\ &(\partial_\tau \{E_1 \sigma_{r+1} + E_{r+1} \sigma_1\}, \partial_\tau P_0 u_{r+1}) \\ &\leq \frac{\eta}{8} \sum_{|\tau| \leq s+1} \|\partial_\tau P_0 u_{r+1}\|^2 + \frac{\alpha}{4} \sum_{|\tau| \leq s} \|\partial_\tau \sigma_{r+1}\|^2 + \frac{\alpha}{8} \sum_{|\tau| \leq s} \|\partial_\tau E_{r+1}\|^2 \\ &\quad + C_{\eta, \alpha} \sum_{|\tau| \leq s+1} \|\partial_\tau \sigma_1\|^4. \end{aligned}$$

In the same fashion, starting with (1.34) and (1.35) respectively, one gets

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\partial_\tau \sigma_{r+1}\|^2 + \alpha \|\nabla \partial_\tau \sigma_{r+1}\|^2 + \alpha \|\partial_\tau \sigma_{r+1}\|^2 \\
&= -(\partial_\tau \{u_1 \cdot \nabla \sigma_{r+1} + P_0 u_{r+1} \cdot \nabla \sigma_1\}, \partial_\tau \sigma_{r+1}) + (\partial_\tau R_{r+1}^\sigma, \partial_\tau \sigma_{r+1}), \\
&\leq \frac{\alpha}{6} \sum_{|\tau| \leq s+1} \|\partial_\tau \sigma_{r+1}\|^2 + \frac{\eta}{8} \sum_{|\tau| \leq s} \|\partial_\tau P_0 u_{r+1}\|^2 + C_\alpha \sum_{|\tau| \leq s+2} \|\partial_\tau \sigma_1\|^4 \\
&\quad + (\partial_\tau R_{r+1}^\sigma, \partial_\tau \sigma_{r+1}); \\
& \frac{1}{2} \frac{d}{dt} \|\partial_\tau \theta_{r+1}\|^2 + \kappa \|\nabla \partial_\tau \theta_{r+1}\|^2 \\
&= -(\partial_\tau \{u_1 \cdot \nabla \theta_{r+1} + P_0 u_{r+1} \cdot \nabla \theta_1\}, \partial_\tau \theta_{r+1}) + (\partial_\tau R_{r+1}^\theta, \partial_\tau \theta_{r+1}) \\
&\leq \frac{\kappa}{2} \sum_{|\tau| \leq s+1} \|\partial_\tau \theta_{r+1}\|^2 + \frac{\eta}{8} \sum_{|\tau| \leq s} \|\partial_\tau P_0 u_{r+1}\|^2 + C_\kappa \sum_{|\tau| \leq s+2} \|\partial_\tau \theta_1\|^4 \\
&\quad + (\partial_\tau R_{r+1}^\theta, \partial_\tau \theta_{r+1}).
\end{aligned}$$

For the electric field, from (1.36), we obtain

$$\|\partial_\tau \nabla E_{r+1}\| \leq \|\partial_\tau \partial_t B_r\| + \|\partial_\tau \sigma_{r+1}\|.$$

The conservation laws (2.1) and (1.38) are utilized to handle no derivative terms: by Poincaré inequality,

$$\begin{aligned}
\|P_0 u_{r+1}\| &\leq C \|\nabla P_0 u_{r+1}\| + C \left| \int P_0 u_{r+1} dx \right| \\
&\leq C \|\nabla P_0 u_{r+1}\| + C \sum_{1 \leq i \leq r} (\|E_i\|^2 + \|B_i\|^2); \\
\|\sigma_{r+1}\| &\leq C \|\nabla \sigma_{r+1}\|; \\
\|\theta_{r+1}\| &\leq C \|\nabla \theta_{r+1}\| + C \sum_{1 \leq i \leq r} (\|E_i\|^2 + \|B_i\|^2); \\
\|E_{r+1}\| &\leq C \|\nabla E_{r+1}\| + C \left| \int E_{r+1} dx \right| \\
&\leq C \|\nabla E_{r+1}\| + C |e^{-\alpha t} \int E_{r+1}^0 dx + \int_0^t e^{-\alpha(t-s)} \ell_r ds|.
\end{aligned}$$

Based on the above estimates, following the argument in [9] (p.30-31), by the induction hypothesis and the Gronwall lemma, one can verify that (2.2) holds for $m = r + 1$ and thus it completes the proof of the theorem. \square

5. UNIFORM SPATIAL ENERGY ESTIMATE

In this section, we consider the following model problem:

$$\begin{aligned}
(5.1) \quad & \partial_t f^\varepsilon + \frac{v}{\varepsilon} \cdot \nabla_x f^\varepsilon + \frac{1}{\varepsilon^2} L f^\varepsilon = h_1^\varepsilon, \\
& \partial_t g^\varepsilon + \frac{v}{\varepsilon} \cdot (\nabla_x g^\varepsilon - \sqrt{\mu} E^\varepsilon) + \frac{1}{\varepsilon^2} \mathcal{L} g^\varepsilon = h_2^\varepsilon,
\end{aligned}$$

coupled with the Maxwell equations

$$(5.2) \quad \begin{aligned} \varepsilon \partial_t E^\varepsilon - \nabla \times B^\varepsilon &= - \int_{\mathbb{R}^3} g^\varepsilon v \sqrt{\mu} dv + j_1^\varepsilon, \quad \nabla \cdot B^\varepsilon = 0, \\ \varepsilon \partial_t B^\varepsilon + \nabla \times E^\varepsilon &= j_2^\varepsilon, \quad \nabla \cdot E^\varepsilon = \int_{\mathbb{R}^3} g^\varepsilon \sqrt{\mu} dv, \end{aligned}$$

where h_1^ε , h_2^ε , j_1^ε and j_2^ε will be given. We shall establish a uniform space-time energy estimate for $f^\varepsilon, g^\varepsilon, E^\varepsilon$ and B^ε .

We use a different representation for the hydrodynamic field (fluid) parts $\mathbf{P}_1 f^\varepsilon$, $\mathbf{P}_2 g^\varepsilon$ in a different way as:

$$[\mathbf{P}_1 f^\varepsilon, \mathbf{P}_2 g^\varepsilon] = [\{a^\varepsilon(t, x) + b^\varepsilon(t, x) \cdot v + c^\varepsilon(t, x)|v|^2\} \sqrt{\mu}, d^\varepsilon(t, x) \sqrt{\mu}].$$

Our goal is to estimate $a^\varepsilon(t, x), b^\varepsilon(t, x), c^\varepsilon(t, x), d^\varepsilon(t, x)$ and $E^\varepsilon(t, x), B^\varepsilon(t, x)$ in terms of $(\mathbf{I} - \mathbf{P}_1) f^\varepsilon$ and $(\mathbf{I} - \mathbf{P}_2) g^\varepsilon$.

We remark that our argument in this section is valid for all $\varepsilon \leq 1$. In particular, the case of $\varepsilon = 1$ yields another proof of the existence of the classical solution for the Vlasov-Maxwell-Boltzmann system without using the time derivative.

Lemma 5.1. *Assume $f^\varepsilon, g^\varepsilon, E^\varepsilon, B^\varepsilon$ are solutions to the system (5.1), (5.2) such that for all $t \geq 0$,*

$$(5.3) \quad \begin{aligned} \int_{\mathbb{T}^3} a^\varepsilon(t, x) dx &= O(\varepsilon(\| [E^\varepsilon, B^\varepsilon] \|^2(t))) + O(\varepsilon \mathcal{A}), \\ \int_{\mathbb{T}^3} b^\varepsilon(t, x) dx &= O(\varepsilon(\| [E^\varepsilon, B^\varepsilon] \|^2(t))) + O(\varepsilon \mathcal{A}), \\ \int_{\mathbb{T}^3} c^\varepsilon(t, x) dx &= O(\varepsilon(\| [E^\varepsilon, B^\varepsilon] \|^2(t))) + O(\varepsilon \mathcal{A}), \\ \int_{\mathbb{T}^3} d^\varepsilon(t, x) dx &= 0 \quad \text{and} \quad \int_{\mathbb{T}^3} B^\varepsilon(t, x) dx = 0, \end{aligned}$$

where we have used the standard big O notation in ε and $\mathcal{A}(t)$ is a function of t . And suppose that $\|E^\varepsilon\|^2 + \|B^\varepsilon\|^2 \leq M$ for some constant M . Then there exists a constant $C_1 > 0$ such that

$$(5.4) \quad \begin{aligned} \sum_{|\gamma| \leq N} \|[\partial_\gamma \mathbf{P}_1 f^\varepsilon, \partial_\gamma \mathbf{P}_2 g^\varepsilon]\|^2 &\leq \varepsilon \frac{dG(t)}{dt} + \frac{C_1}{\varepsilon^2} \sum_{|\gamma| \leq N} \|[\partial_\gamma \{\mathbf{I} - \mathbf{P}_1\} f^\varepsilon, \partial_\gamma \{\mathbf{I} - \mathbf{P}_2\} g^\varepsilon]\|_\nu^2 \\ &+ C_1 \varepsilon^2 \sum_{|\gamma| \leq N-1} \|\partial_\gamma h_{||}^\varepsilon\|^2 + \varepsilon^2 \mathcal{A}^2 + C_1 \varepsilon^2 (\|j_1^\varepsilon\|^2 + \|\nabla j_1^\varepsilon\|^2 + \|j_2^\varepsilon\|^2 + \|\nabla j_2^\varepsilon\|^2) \end{aligned}$$

where $\varepsilon \leq 1$ and $G(t)$ is a function of t satisfying

$$(5.5) \quad |G(t)| \leq \sum_{|\gamma| \leq N} \|[\partial_\gamma f^\varepsilon, \partial_\gamma g^\varepsilon]\|^2(t) + \sum_{|\gamma| \leq N} \|[\partial_\gamma E^\varepsilon, \partial_\gamma B^\varepsilon]\|^2(t),$$

and $\partial_\gamma h_{||}^\varepsilon$ is the L_v^2 projection of $[\partial_\gamma h_1^\varepsilon, \partial_\gamma h_2^\varepsilon](t, x, v)$ on the subspace generated by

$$[\sqrt{\mu}, v_i \sqrt{\mu}, v_i v_j \sqrt{\mu}, v_i |v|^2 \sqrt{\mu}].$$

Proof. As well illustrated in [9], there are two fundamental ingredients in the proof. First of all, we use the *Local Conservation Laws*: By multiplying $\sqrt{\mu}, v\sqrt{\mu}, |v|^2\sqrt{\mu}$ with the first equation of (5.1), $\sqrt{\mu}$ with the second equation and then integrating in $v \in \mathbb{R}^3$, we obtain the following:

$$\begin{aligned}
 (5.6) \quad & a_t^\varepsilon = \frac{1}{2\varepsilon} \langle v \cdot \nabla_x (\mathbf{I} - \mathbf{P}_1) f^\varepsilon, |v|^2 \sqrt{\mu} \rangle + \langle h_1^\varepsilon, \left\{ \frac{5}{2} - \frac{|v|^2}{2} \right\} \sqrt{\mu} \rangle, \\
 & c_t^\varepsilon + \frac{1}{3\varepsilon} \nabla_x \cdot b^\varepsilon = -\frac{1}{6\varepsilon} \langle v \cdot \nabla_x (\mathbf{I} - \mathbf{P}_1) f^\varepsilon, |v|^2 \sqrt{\mu} \rangle + \langle h_1^\varepsilon, \left\{ \frac{|v|^2}{6} - \frac{1}{2} \right\} \sqrt{\mu} \rangle, \\
 & b_t^\varepsilon + \frac{1}{\varepsilon} \{ \nabla_x a^\varepsilon + 5 \nabla_x c^\varepsilon \} = -\frac{1}{\varepsilon} \langle v \cdot \nabla_x (\mathbf{I} - \mathbf{P}_1) f^\varepsilon, v \sqrt{\mu} \rangle + \langle h_1^\varepsilon, v \sqrt{\mu} \rangle, \\
 & d_t^\varepsilon = -\frac{1}{\varepsilon} \langle v \cdot \nabla_x (\mathbf{I} - \mathbf{P}_2) g^\varepsilon, \sqrt{\mu} \rangle + \langle h_2^\varepsilon, \sqrt{\mu} \rangle.
 \end{aligned}$$

The second ingredient is the study of the *Macroscopic Equations*: notice that by plugging $f^\varepsilon \equiv \mathbf{P}_1 f^\varepsilon + (\mathbf{I} - \mathbf{P}_1) f^\varepsilon$ and $g^\varepsilon \equiv \mathbf{P}_2 g^\varepsilon + (\mathbf{I} - \mathbf{P}_2) g^\varepsilon$ into (5.1),

$$\begin{aligned}
 & \varepsilon \{ a_t^\varepsilon + b_t^\varepsilon \cdot v + c_t^\varepsilon |v|^2 \} \sqrt{\mu} + v \cdot \{ \nabla_x a^\varepsilon + \nabla_x b^\varepsilon \cdot v + \nabla_x c^\varepsilon |v|^2 \} \sqrt{\mu} \\
 & = -\{ \varepsilon \partial_t + v \cdot \nabla_x \} (\mathbf{I} - \mathbf{P}_1) f^\varepsilon - \frac{1}{\varepsilon} L(\mathbf{I} - \mathbf{P}_1) f^\varepsilon + \varepsilon h_1^\varepsilon, \\
 & \varepsilon d_t^\varepsilon \sqrt{\mu} + v \cdot \{ \nabla_x d^\varepsilon - E^\varepsilon \} \sqrt{\mu} \\
 & = -\{ \varepsilon \partial_t + v \cdot \nabla_x \} (\mathbf{I} - \mathbf{P}_2) g^\varepsilon - \frac{1}{\varepsilon} \mathcal{L}(\mathbf{I} - \mathbf{P}_2) g^\varepsilon + \varepsilon h_2^\varepsilon.
 \end{aligned}$$

Fixing t and x , and comparing the coefficients on both sides in front of the $[\sqrt{\mu}, v\sqrt{\mu}, v_i v_j \sqrt{\mu}, v_i |v|^2 \sqrt{\mu}]$, we obtain the *macroscopic* equations as

$$(5.7) \quad \nabla_x c^\varepsilon = l_c^\varepsilon + \varepsilon h_c^\varepsilon,$$

$$(5.8) \quad \varepsilon \partial_t c^\varepsilon + \partial_i b_i^\varepsilon = l_i^\varepsilon + \varepsilon h_i^\varepsilon,$$

$$(5.9) \quad \partial_i b_j^\varepsilon + \partial_j b_i^\varepsilon = l_{ij}^\varepsilon + \varepsilon h_{ij}^\varepsilon \quad \text{for } i \neq j,$$

$$(5.10) \quad \varepsilon \partial_t b_i^\varepsilon + \partial_i a^\varepsilon = l_{bi}^\varepsilon + \varepsilon h_{bi}^\varepsilon,$$

$$(5.11) \quad \varepsilon \partial_t a^\varepsilon = l_a^\varepsilon + \varepsilon h_a^\varepsilon,$$

$$(5.12) \quad \varepsilon \partial_t d^\varepsilon = l_d^\varepsilon + \varepsilon h_d^\varepsilon,$$

$$(5.13) \quad \nabla_x d^\varepsilon - E^\varepsilon = l_e^\varepsilon + \varepsilon h_e^\varepsilon.$$

Here by elementary linear algebra, the linear parts $l_c^\varepsilon, l_i^\varepsilon, l_{ij}^\varepsilon, l_{bi}^\varepsilon, l_a^\varepsilon, l_d^\varepsilon$ and l_e^ε are all of the form

$$\begin{aligned}
 (5.14) \quad & \text{either } \langle -\{ \varepsilon \partial_t + v \cdot \nabla_x \} (\mathbf{I} - \mathbf{P}_1) f^\varepsilon, \zeta \rangle - \frac{1}{\varepsilon} \langle L(\mathbf{I} - \mathbf{P}_1) f^\varepsilon, \zeta \rangle \\
 & \text{or } \langle -\{ \varepsilon \partial_t + v \cdot \nabla_x \} (\mathbf{I} - \mathbf{P}_2) g^\varepsilon, \zeta \rangle - \frac{1}{\varepsilon} \langle \mathcal{L}(\mathbf{I} - \mathbf{P}_2) g^\varepsilon, \zeta \rangle
 \end{aligned}$$

where ζ is a (different) linear combination of the basis $[\sqrt{\mu}, v\sqrt{\mu}, v_i v_j \sqrt{\mu}, v_i |v|^2 \sqrt{\mu}]$ accordingly, while $h_c^\varepsilon, h_i^\varepsilon, h_{ij}^\varepsilon, h_{bi}^\varepsilon, h_a^\varepsilon, h_d^\varepsilon$ and h_e^ε are defined as $\langle h^\varepsilon, \zeta \rangle$ with same choices of ζ . Notice that

$$||\partial_\gamma h_c^\varepsilon|| + ||\partial_\gamma h_i^\varepsilon|| + ||\partial_\gamma h_{ij}^\varepsilon|| + ||\partial_\gamma h_{bi}^\varepsilon|| + ||\partial_\gamma h_a^\varepsilon|| + ||\partial_\gamma h_d^\varepsilon|| + ||\partial_\gamma h_e^\varepsilon|| \leq C ||\partial_\gamma h||.$$

The macroscopic equations (5.7)-(5.11) have the same structure as the pure Boltzmann case. Following the proof of Lemma 6.1 in [9], we can deduce the

estimates on $\nabla_x \partial_{\gamma_1} a^\varepsilon$, $\nabla_x \partial_{\gamma_1} b^\varepsilon$ and $\nabla_x \partial_{\gamma_1} c^\varepsilon$: for $|\gamma_1| \leq N-1$,

$$\begin{aligned}
\frac{1}{2} \|\nabla \partial_{\gamma_1} b^\varepsilon\|^2 &\leq -\frac{d}{dt} \int_{\mathbb{T}^3} \langle \varepsilon(\mathbf{I} - \mathbf{P}_1) \partial_{\gamma_1} f^\varepsilon, \zeta_{ij} \rangle \cdot \partial_j \partial_{\gamma_1} b^\varepsilon dx \\
&\quad + \frac{C}{\varepsilon^2} \{ \|\nabla_x \partial_{\gamma_1} (\mathbf{I} - \mathbf{P}_1) f^\varepsilon\|_\nu^2 + \|\partial_{\gamma_1} (\mathbf{I} - \mathbf{P}_1) f^\varepsilon\|_\nu^2 \} \\
&\quad + C\varepsilon^2 \{ \|\nabla_x \partial_{\gamma_1} a^\varepsilon\|^2 + \|\nabla_x \partial_{\gamma_1} c^\varepsilon\|^2 + \|\partial_{\gamma_1} h_\parallel^\varepsilon\|^2 \}, \\
\frac{1}{2} \|\nabla \partial_{\gamma_1} c^\varepsilon\|^2 &\leq -\frac{d}{dt} \int_{\mathbb{T}^3} \langle \varepsilon(\mathbf{I} - \mathbf{P}_1) \partial_{\gamma_1} f^\varepsilon, \zeta_c \rangle \cdot \nabla_x \partial_{\gamma_1} c^\varepsilon dx \\
&\quad + \frac{C}{\varepsilon^2} \{ \|\nabla_x \partial_{\gamma_1} (\mathbf{I} - \mathbf{P}_1) f^\varepsilon\|_\nu^2 + \|\partial_{\gamma_1} (\mathbf{I} - \mathbf{P}_1) f^\varepsilon\|_\nu^2 \} \\
&\quad + C\varepsilon^2 \{ \|\nabla_x \partial_{\gamma_1} a^\varepsilon\|^2 + \|\nabla_x \partial_{\gamma_1} c^\varepsilon\|^2 + \|\partial_{\gamma_1} h_\parallel^\varepsilon\|^2 \}, \\
\frac{1}{2} \|\nabla \partial_{\gamma_1} a^\varepsilon\|^2 &\leq -\frac{d}{dt} \int_{\mathbb{T}^3} \langle \varepsilon(\mathbf{I} - \mathbf{P}_1) \partial_{\gamma_1} f^\varepsilon, \zeta \rangle \cdot \nabla_x \partial_{\gamma_1} a^\varepsilon dx + \int_{\mathbb{T}^3} \varepsilon \partial_{\gamma_1} b^\varepsilon \cdot \nabla_x \partial_{\gamma_1} a^\varepsilon dx \\
&\quad + \frac{C}{\varepsilon^2} \{ \|\nabla_x \partial_{\gamma_1} (\mathbf{I} - \mathbf{P}_1) f^\varepsilon\|_\nu^2 + \|\partial_{\gamma_1} (\mathbf{I} - \mathbf{P}_1) f^\varepsilon\|_\nu^2 \} \\
&\quad + C\varepsilon^2 \{ \|\partial_{\gamma_1} h_\parallel^\varepsilon\|^2 + \|\nabla \partial_{\gamma_1} b^\varepsilon\|^2 \}.
\end{aligned}$$

We shall, however, estimate $\nabla_x \partial_{\gamma_1} d^\varepsilon$ in the same spirit.

$$\begin{aligned}
\Delta \partial_{\gamma_1} d_i^\varepsilon &= \sum_j \partial_{jj} \partial_{\gamma_1} d_i^\varepsilon \\
&= \sum_j \partial_j [\partial_{\gamma_1} E_j^\varepsilon + \partial_{\gamma_1} l_e^\varepsilon + \varepsilon \partial_{\gamma_1} h_e^\varepsilon] \text{ by (5.13)} \\
&= \partial_{\gamma_1} d^\varepsilon + \sum_j [\partial_j \partial_{\gamma_1} l_e^\varepsilon + \varepsilon \partial_j \partial_{\gamma_1} h_e^\varepsilon] \text{ (since } \nabla \cdot E^\varepsilon = d^\varepsilon)
\end{aligned}$$

Multiply the above by $\partial_{\gamma_1} d^\varepsilon$ and integrate to get

$$(5.15) \quad \int_{\mathbb{T}^3} |\nabla_x \partial_{\gamma_1} d^\varepsilon|^2 + |\partial_{\gamma_1} d^\varepsilon|^2 dx = \sum_j \int_{\mathbb{T}^3} (\partial_{\gamma_1} l_e^\varepsilon + \varepsilon \partial_{\gamma_1} h_e^\varepsilon) \cdot \partial_j \partial_{\gamma_1} d^\varepsilon dx.$$

Recall that

$$\begin{aligned}
\partial_{\gamma_1} l_e^\varepsilon &= -\langle \varepsilon \partial_t (\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_1} g^\varepsilon, v\sqrt{\mu} \rangle \\
&\quad - \langle v \cdot \nabla_x (\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_1} g^\varepsilon, v\sqrt{\mu} \rangle - \frac{1}{\varepsilon} \langle \mathcal{L}(\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_1} g^\varepsilon, v\sqrt{\mu} \rangle.
\end{aligned}$$

Last two terms are of the desired form. As for the first one, we integrate it by parts in the t variable:

$$\begin{aligned}
&\int_{\mathbb{T}^3} \langle \varepsilon \partial_t (\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_1} g^\varepsilon, v\sqrt{\mu} \rangle \cdot \partial_j \partial_{\gamma_1} d^\varepsilon dx \\
&= \frac{d}{dt} \int_{\mathbb{T}^3} \langle \varepsilon (\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_1} g^\varepsilon, v\sqrt{\mu} \rangle \cdot \partial_j \partial_{\gamma_1} d^\varepsilon dx \\
&\quad + \int_{\mathbb{T}^3} \langle (\mathbf{I} - \mathbf{P}_2) \partial_j \partial_{\gamma_1} g^\varepsilon, v\sqrt{\mu} \rangle \cdot \varepsilon \partial_t \partial_{\gamma_1} d^\varepsilon dx.
\end{aligned}$$

Replace $\varepsilon \partial_t \partial_{\gamma_1} d^\varepsilon$ with $\partial_{\gamma_1} l_d^\varepsilon + \varepsilon \partial_{\gamma_1} h_d^\varepsilon$ in the latter integral by the macroscopic equation (5.12). Thus (5.15) leads to the following:

$$(5.16) \quad \begin{aligned} \frac{1}{2}(\|\nabla \partial_{\gamma_1} d^\varepsilon\|^2 + \|\partial_{\gamma_1} d^\varepsilon\|^2) &\leq -\frac{d}{dt} \int_{\mathbb{T}^3} \langle \varepsilon(\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_1} g^\varepsilon, v\sqrt{\mu} \rangle \cdot \partial_j \partial_{\gamma_1} d^\varepsilon dx \\ &+ \frac{C}{\varepsilon^2} \{ \|\nabla_x \partial_{\gamma_1} (\mathbf{I} - \mathbf{P}_2) g^\varepsilon\|_\nu^2 + \|\partial_{\gamma_1} (\mathbf{I} - \mathbf{P}_2) g^\varepsilon\|_\nu^2 \} \\ &+ C\varepsilon^2 \|\partial_{\gamma_1} h_\parallel^\varepsilon\|^2. \end{aligned}$$

The field estimate, the new ingredient of the proof, comes into play for the need of the estimates on no derivative terms a^ε , b^ε and c^ε to complete the proof of Lemma: from (5.3) we obtain by the Poincaré inequality,

$$(5.17) \quad \begin{aligned} \|a^\varepsilon\|^2 + \|b^\varepsilon\|^2 + \|c^\varepsilon\|^2 &\leq \|\nabla a^\varepsilon\|^2 + \|\nabla b^\varepsilon\|^2 + \|\nabla c^\varepsilon\|^2 \\ &+ \varepsilon^2 (\|E^\varepsilon\|^2 + \|B^\varepsilon\|^2) + \varepsilon^2 \mathcal{A}^2. \end{aligned}$$

Also it will play an important role not only to close the energy estimates in later sections but also to achieve decay rates (2.2) and (2.8). The intriguing computation is carried out in two steps: firstly, the electric field will be estimated via the macroscopic equation (5.13) and then the Maxwell equations (5.2) will give rise to the estimate on the magnetic field.

We start with (5.13): $E^\varepsilon = \nabla_x d^\varepsilon - l_e^\varepsilon - \varepsilon h_e^\varepsilon$.

$$(5.18) \quad \begin{aligned} \|\partial_{\gamma_1} E^\varepsilon\|^2 &= \int_{\mathbb{T}^3} (\nabla_x \partial_{\gamma_1} d^\varepsilon - \partial_{\gamma_1} l_e^\varepsilon - \varepsilon \partial_{\gamma_1} h_e^\varepsilon) \cdot \partial_{\gamma_1} E^\varepsilon dx \\ &\leq \int_{\mathbb{T}^3} (\nabla_x \partial_{\gamma_1} d^\varepsilon - \varepsilon \partial_{\gamma_1} h_e^\varepsilon) \cdot \partial_{\gamma_1} E^\varepsilon dx - \int_{\mathbb{T}^3} \partial_{\gamma_1} l_e^\varepsilon \cdot \partial_{\gamma_1} E^\varepsilon dx \\ &\leq -\|\partial_{\gamma_1} d^\varepsilon\|^2 + \frac{\|\partial_{\gamma_1} E^\varepsilon\|^2}{4} + \varepsilon^2 \|\partial_{\gamma_1} h_e^\varepsilon\|^2 - \int_{\mathbb{T}^3} \partial_{\gamma_1} l_e^\varepsilon \cdot \partial_{\gamma_1} E^\varepsilon dx \end{aligned}$$

The last term needs a special care.

$$\begin{aligned} - \int_{\mathbb{T}^3} \partial_{\gamma_1} l_e^\varepsilon \cdot \partial_{\gamma_1} E^\varepsilon dx &= \int_{\mathbb{T}^3} \langle \varepsilon \partial_t \partial_{\gamma_1} (\mathbf{I} - \mathbf{P}_2) g^\varepsilon, v\sqrt{\mu} \rangle \cdot \partial_{\gamma_1} E^\varepsilon dx \\ &+ \int_{\mathbb{T}^3} \langle v \cdot \nabla_x \partial_{\gamma_1} (\mathbf{I} - \mathbf{P}_2) g^\varepsilon + \frac{1}{\varepsilon} \mathcal{L}(\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_1} g^\varepsilon, v\sqrt{\mu} \rangle \cdot \partial_{\gamma_1} E^\varepsilon dx \\ &\equiv (I) + (II) \end{aligned}$$

It is easy to see that the second term (II) is bounded by

$$\frac{\|\partial_{\gamma_1} E^\varepsilon\|^2}{4} + \frac{C}{\varepsilon^2} \{ \|\nabla_x (\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_1} g^\varepsilon\|_\nu^2 + \|(\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_1} g^\varepsilon\|_\nu^2 \}.$$

As for the first term (I) , we first integrate it by parts in t :

$$\frac{d}{dt} \int_{\mathbb{T}^3} \langle \varepsilon(\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_1} g^\varepsilon, v\sqrt{\mu} \rangle \cdot \partial_{\gamma_1} E^\varepsilon dx - \int_{\mathbb{T}^3} \langle (\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_1} g^\varepsilon, v\sqrt{\mu} \rangle \cdot \varepsilon \partial_t \partial_{\gamma_1} E^\varepsilon dx.$$

By using the Maxwell equation (5.2) to eliminate $\varepsilon \partial_t \partial_{\gamma_1} E^\varepsilon$ in the second term, we get the following:

$$\begin{aligned}
& - \int_{\mathbb{T}^3} \langle (\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_1} g^\varepsilon, v \sqrt{\mu} \rangle \cdot \varepsilon \partial_t \partial_{\gamma_1} E^\varepsilon dx \\
& = \int_{\mathbb{T}^3} |\langle (\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_1} g^\varepsilon, v \sqrt{\mu} \rangle|^2 dx + \int_{\mathbb{T}^3} \langle (\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_1} g^\varepsilon, v \sqrt{\mu} \rangle \cdot \partial_{\gamma_1} j_1^\varepsilon dx \\
& \quad - \int_{\mathbb{T}^3} \langle (\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_1} g^\varepsilon, v \sqrt{\mu} \rangle \cdot \nabla \times \partial_{\gamma_1} B^\varepsilon dx \\
& \leq \frac{3}{2} \|(\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_1} g^\varepsilon\|_\nu^2 + \frac{1}{2} \|\partial_{\gamma_1} j_1^\varepsilon\|^2 + C_\xi \|\nabla_x (\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_1} g^\varepsilon\|_\nu^2 + \xi \|\partial_{\gamma_1} B^\varepsilon\|^2,
\end{aligned}$$

where ξ is a fixed small number and we have integrated by parts in x to get the last inequality. Hence (5.18) leads to

$$\begin{aligned}
(5.19) \quad & \frac{1}{2} \|\partial_{\gamma_1} E^\varepsilon\|^2 + \|\partial_{\gamma_1} d^\varepsilon\|^2 \leq \frac{d}{dt} \int_{\mathbb{T}^3} \langle \varepsilon (\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_1} g^\varepsilon, v \sqrt{\mu} \rangle \cdot \partial_{\gamma_1} E^\varepsilon dx \\
& \quad + C_\xi \{ \|\nabla_x (\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_1} g^\varepsilon\|_\nu^2 + \|(\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_1} g^\varepsilon\|_\nu^2 \} \\
& \quad + \varepsilon^2 \|\partial_{\gamma_1} h_\varepsilon^\varepsilon\|^2 + \frac{1}{2} \|\partial_{\gamma_1} j_1^\varepsilon\|^2 + \xi \|\partial_{\gamma_1} B^\varepsilon\|^2.
\end{aligned}$$

As for the estimate on $\partial_{\gamma_1} B^\varepsilon$, recall that $\nabla \times B^\varepsilon = \varepsilon \partial_t E^\varepsilon + \langle (\mathbf{I} - \mathbf{P}_2) g^\varepsilon, v \sqrt{\mu} \rangle - j_1^\varepsilon$. Let $|\gamma_2| \leq N - 2$.

$$\begin{aligned}
\|\nabla \times \partial_{\gamma_2} B^\varepsilon\|^2 & = \int_{\mathbb{T}^3} (\varepsilon \partial_t \partial_{\gamma_2} E^\varepsilon + \langle (\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_2} g^\varepsilon, v \sqrt{\mu} \rangle - \partial_{\gamma_2} j_1^\varepsilon) \cdot \nabla \times \partial_{\gamma_2} B^\varepsilon dx \\
& = \frac{d}{dt} \int_{\mathbb{T}^3} \varepsilon \partial_{\gamma_2} E^\varepsilon \cdot \nabla \times \partial_{\gamma_2} B^\varepsilon dx - \int_{\mathbb{T}^3} \partial_{\gamma_2} E^\varepsilon \cdot \nabla \times \varepsilon \partial_t \partial_{\gamma_2} B^\varepsilon dx \\
& \quad + \int_{\mathbb{T}^3} (\langle (\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_2} g^\varepsilon, v \sqrt{\mu} \rangle - \partial_{\gamma_2} j_1^\varepsilon) \cdot \nabla \times \partial_{\gamma_2} B^\varepsilon dx
\end{aligned}$$

By (5.2), the second term containing $\varepsilon \partial_t \partial_{\gamma_2} B^\varepsilon$ in the above can be majorized as following:

$$\begin{aligned}
& \int_{\mathbb{T}^3} \partial_{\gamma_2} E^\varepsilon \cdot (\nabla \times \nabla \times \partial_{\gamma_2} E^\varepsilon - \nabla \times \partial_{\gamma_2} j_2^\varepsilon) dx \\
& = \int_{\mathbb{T}^3} \partial_{\gamma_2} E^\varepsilon \cdot (\nabla (\nabla \cdot \partial_{\gamma_2} E^\varepsilon) - \Delta \partial_{\gamma_2} E^\varepsilon - \nabla \times \partial_{\gamma_2} j_2^\varepsilon) dx \\
& = \int_{\mathbb{T}^3} \partial_{\gamma_2} E^\varepsilon \cdot \nabla \partial_{\gamma_2} d^\varepsilon dx + \int_{\mathbb{T}^3} |\nabla \partial_{\gamma_2} E^\varepsilon|^2 dx - \int_{\mathbb{T}^3} \partial_{\gamma_2} E^\varepsilon \cdot \nabla \times \partial_{\gamma_2} j_2^\varepsilon dx \\
& \leq \int_{\mathbb{T}^3} |\partial_{\gamma_2} E^\varepsilon|^2 + |\nabla \partial_{\gamma_2} E^\varepsilon|^2 dx + \frac{1}{2} \int_{\mathbb{T}^3} |\nabla \partial_{\gamma_2} d^\varepsilon|^2 + |\nabla \times \partial_{\gamma_2} j_2^\varepsilon|^2 dx
\end{aligned}$$

On the other hand, the last term is bounded by

$$C_\xi (\|(\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_2} g^\varepsilon\|_\nu^2 + \|\partial_{\gamma_2} j_1^\varepsilon\|^2) + \xi \|\nabla \times \partial_{\gamma_2} B^\varepsilon\|^2$$

for any small number ξ . After absorbing $\xi \|\nabla \times \partial_{\gamma_2} B^\varepsilon\|^2$ into the LHS, we obtain for some $0 < C_2 = 1 - \xi < 1$,

$$\begin{aligned}
(5.20) \quad & C_2 \|\nabla \times \partial_{\gamma_2} B^\varepsilon\|^2 \leq \frac{d}{dt} \int_{\mathbb{T}^3} \varepsilon \partial_{\gamma_2} E^\varepsilon \cdot \nabla \times \partial_{\gamma_2} B^\varepsilon dx + \int_{\mathbb{T}^3} |\partial_{\gamma_2} E^\varepsilon|^2 + |\nabla \partial_{\gamma_2} E^\varepsilon|^2 dx \\
& \quad + \frac{1}{2} \int_{\mathbb{T}^3} |\nabla \partial_{\gamma_2} d^\varepsilon|^2 + |\nabla \times \partial_{\gamma_2} j_2^\varepsilon|^2 dx + C_\xi (\|(\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_2} g^\varepsilon\|_\nu^2 + \|\partial_{\gamma_2} j_1^\varepsilon\|^2).
\end{aligned}$$

Letting $|\gamma_2| = |\gamma_1| - 1 \geq 0$, combine (5.19) with (5.20) to get for some $0 < C_3 < 1$,

$$(5.21) \quad \begin{aligned} C_3 \|\partial_{\gamma_1} E^\varepsilon\|^2 &\leq \frac{d}{dt} \int_{\mathbb{T}^3} \langle \varepsilon(\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_1} g^\varepsilon, v\sqrt{\mu} \rangle \cdot \partial_{\gamma_1} E^\varepsilon + \xi C_2 \varepsilon \partial_{\gamma_2} E^\varepsilon \cdot \nabla \times \partial_{\gamma_2} B^\varepsilon dx \\ &+ C_\xi \{ \|\nabla_x(\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_1} g^\varepsilon\|_\nu^2 + \|(\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_1} g^\varepsilon\|_\nu^2 + \|(\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_2} g^\varepsilon\|_\nu^2 \} \\ &+ \varepsilon^2 \|\partial_{\gamma_1} h_e^\varepsilon\|^2 + \|\partial_{\gamma_1} j_1^\varepsilon\|^2 + \xi \|\nabla \times \partial_{\gamma_2} j_2^\varepsilon\|^2. \end{aligned}$$

Now let us go back to (5.17). After applying (5.20) and (5.21) for $|\gamma_1| = 0, 1$ and $|\gamma_2| = 0$ with fixed small ξ , we have

$$(5.22) \quad \begin{aligned} C_4 (\|E^\varepsilon\|^2 + \|B^\varepsilon\|^2) &\leq \frac{d}{dt} \int_{\mathbb{T}^3} \varepsilon E^\varepsilon \cdot \nabla \times B^\varepsilon dx \\ &+ \frac{d}{dt} \int_{\mathbb{T}^3} \langle \varepsilon(\mathbf{I} - \mathbf{P}_2) \nabla_x g^\varepsilon, v\sqrt{\mu} \rangle \cdot \nabla E^\varepsilon + \langle \varepsilon(\mathbf{I} - \mathbf{P}_2) g^\varepsilon, v\sqrt{\mu} \rangle \cdot E^\varepsilon dx \\ &+ \frac{C}{\varepsilon^2} \{ \|\nabla_x(\mathbf{I} - \mathbf{P}_2) \nabla_x g^\varepsilon\|_\nu^2 + \|(\mathbf{I} - \mathbf{P}_2) \nabla_x g^\varepsilon\|_\nu^2 + \|(\mathbf{I} - \mathbf{P}_2) g^\varepsilon\|_\nu^2 \} \\ &+ \varepsilon^2 (\|\nabla_x h_e^\varepsilon\|^2 + \|h_e^\varepsilon\|^2) + \|j_1^\varepsilon\|^2 + \|\nabla j_1^\varepsilon\|^2 + \|j_2^\varepsilon\|^2 + \|\nabla j_2^\varepsilon\|^2, \end{aligned}$$

where $0 < C_4 < 1$ and $\varepsilon \leq 1$. This estimate with (5.17) finally finishes (5.4) and thus Lemma. \square

Lemma 5.2. *Assume $f^\varepsilon, g^\varepsilon, E^\varepsilon, B^\varepsilon$ are solutions to the system (5.1), (5.2) and satisfy (5.3). Then there exists constant $C_1 \geq 1$ such that the following energy estimate is valid:*

$$(5.23) \quad \begin{aligned} &\frac{d}{dt} \{ C_1 \sum_{|\gamma| \leq N} \{ \|[\partial_\gamma f^\varepsilon, \partial_\gamma g^\varepsilon]\|^2 + \|[\partial_\gamma E^\varepsilon, \partial_\gamma B^\varepsilon]\|^2 \} - \varepsilon \delta G(t) \} \\ &+ \delta \sum_{|\gamma| \leq N} \{ \frac{1}{\varepsilon^2} \|[\partial_\gamma \{\mathbf{I} - \mathbf{P}_1\} f^\varepsilon, \partial_\gamma \{\mathbf{I} - \mathbf{P}_2\} g^\varepsilon]\|_\nu^2 + \|[\partial_\gamma \mathbf{P}_1 f^\varepsilon, \partial_\gamma \mathbf{P}_2 g^\varepsilon]\|^2 \} \\ &\leq 2C_1 \sum_{|\gamma| \leq N} \{ (\partial_\gamma h_1^\varepsilon, \partial_\gamma f^\varepsilon) + (\partial_\gamma h_2^\varepsilon, \partial_\gamma g^\varepsilon) \} + \varepsilon^2 \delta \sum_{|\gamma| \leq N-1} \|\partial_\gamma h_1^\varepsilon\|^2 \\ &+ \frac{2C_1}{\varepsilon} \sum_{|\gamma| \leq N} (\|\partial_\gamma E^\varepsilon\| \cdot \|\partial_\gamma j_1^\varepsilon\| + \|\partial_\gamma B^\varepsilon\| \cdot \|\partial_\gamma j_2^\varepsilon\|) \\ &+ \varepsilon^2 \delta \mathcal{A}^2 + \varepsilon^2 \delta (\|j_1^\varepsilon\|^2 + \|\nabla j_1^\varepsilon\|^2 + \|j_2^\varepsilon\|^2 + \|\nabla j_2^\varepsilon\|^2). \end{aligned}$$

Proof. The standard ∂_γ energy estimates with (5.1) and (5.2) give rise to

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \{ \|\partial_\gamma f^\varepsilon\|^2 + \|\partial_\gamma g^\varepsilon\|^2 \} + \frac{1}{\varepsilon^2} \{ (L \partial_\gamma f^\varepsilon, \partial_\gamma f^\varepsilon) + (\mathcal{L} \partial_\gamma g^\varepsilon, \partial_\gamma g^\varepsilon) \} \\ &- \frac{1}{\varepsilon} (v\sqrt{\mu} \cdot \partial_\gamma E^\varepsilon, \partial_\gamma g^\varepsilon) = (\partial_\gamma h_1^\varepsilon, \partial_\gamma f^\varepsilon) + (\partial_\gamma h_2^\varepsilon, \partial_\gamma g^\varepsilon). \end{aligned}$$

Use (5.2) twice to deal with $-\frac{1}{\varepsilon} (v\sqrt{\mu} \cdot \partial_\gamma E^\varepsilon, \partial_\gamma g^\varepsilon)$:

$$\begin{aligned} &-\frac{1}{\varepsilon} (v\sqrt{\mu} \cdot \partial_\gamma E^\varepsilon, \partial_\gamma g^\varepsilon) = \frac{1}{\varepsilon} \int_{\mathbb{T}^3} \partial_\gamma E^\varepsilon \cdot \{ \varepsilon \partial_t \partial_\gamma E^\varepsilon - \nabla \times \partial_\gamma B^\varepsilon - \partial_\gamma j_1^\varepsilon \} dx \\ &= \frac{1}{2} \frac{d}{dt} \{ \|\partial_\gamma E^\varepsilon\|^2 + \|\partial_\gamma B^\varepsilon\|^2 \} - \frac{1}{\varepsilon} \int_{\mathbb{T}^3} \{ \partial_\gamma E^\varepsilon \cdot \partial_\gamma j_1^\varepsilon + \partial_\gamma B^\varepsilon \cdot \partial_\gamma j_2^\varepsilon \} dx \end{aligned}$$

But from Lemma 5.1, there is a constant $C_1 \geq 1$ such that

$$\begin{aligned} & \frac{\delta}{2\varepsilon^2} \sum_{|\gamma| \leq N} \|[\partial_\gamma \{\mathbf{I} - \mathbf{P}_1\} f^\varepsilon, \partial_\gamma \{\mathbf{I} - \mathbf{P}_2\} g^\varepsilon]\|_\nu^2 \\ & \geq \frac{1}{2C_1} \sum_{|\gamma| \leq N} \{\delta \|[\partial_\gamma \mathbf{P}_1 f^\varepsilon, \partial_\gamma \mathbf{P}_2 g^\varepsilon]\| - \varepsilon \delta G(t)\} - \varepsilon^2 \delta \sum_{|\gamma| \leq N-1} \|\partial_\gamma h_\parallel^\varepsilon\|^2 \\ & \quad - \varepsilon^2 \delta \mathcal{A}^2 - \varepsilon^2 \delta (\|j_1^\varepsilon\|^2 + \|\nabla j_1^\varepsilon\|^2 + \|j_2^\varepsilon\|^2 + \|\nabla j_2^\varepsilon\|^2). \end{aligned}$$

By (1.20), multiplying by C_1 and collecting terms, we deduce our lemma. \square

6. THE FIRST ORDER REMAINDER

In this section, we prove Theorem 2.2. We study the solution to the kinetic equation (2.3). We first establish the spatial energy estimate for $\partial_\gamma f^\varepsilon$ and $\partial_\gamma g^\varepsilon$.

Lemma 6.1. *Let $f^\varepsilon, g^\varepsilon, E^\varepsilon, B^\varepsilon$ be solutions to the system (2.3)-(2.4) which satisfy the following conservation laws:*

$$\begin{aligned} (6.1) \quad & \int_{\mathbb{T}^3} a^\varepsilon(t, x) dx = \frac{\varepsilon}{2} \int_{\mathbb{T}^3} |E^\varepsilon(t, x)|^2 + |B^\varepsilon(t, x)|^2 dx, \\ & \int_{\mathbb{T}^3} b^\varepsilon(t, x) dx = -\varepsilon \int_{\mathbb{T}^3} E^\varepsilon(t, x) \times B^\varepsilon(t, x) dx, \\ & \int_{\mathbb{T}^3} c^\varepsilon(t, x) dx = -\frac{\varepsilon}{6} \int_{\mathbb{T}^3} |E^\varepsilon(t, x)|^2 + |B^\varepsilon(t, x)|^2 dx, \\ & \int_{\mathbb{T}^3} d^\varepsilon(t, x) dx = 0 \quad \text{and} \quad \int_{\mathbb{T}^3} B^\varepsilon(t, x) dx = 0. \end{aligned}$$

Then for any instant energy functional $\mathcal{E}_N(f^\varepsilon, g^\varepsilon, E^\varepsilon, B^\varepsilon)$ where $N \geq 8$, there is a constant $C > 0$ such that

$$\begin{aligned} (6.2) \quad & \frac{d}{dt} \{C_1 \{ \|[\partial_\gamma f^\varepsilon, \partial_\gamma g^\varepsilon]\|^2 + \|[\partial_\gamma E^\varepsilon, \partial_\gamma B^\varepsilon]\|^2 \} - \varepsilon \delta G(t) \} \\ & + \delta \left\{ \frac{1}{\varepsilon^2} \|[\partial_\gamma \{\mathbf{I} - \mathbf{P}_1\} f^\varepsilon, \partial_\gamma \{\mathbf{I} - \mathbf{P}_2\} g^\varepsilon]\|_\nu^2 + \|[\partial_\gamma \mathbf{P}_1 f^\varepsilon, \partial_\gamma \mathbf{P}_2 g^\varepsilon]\|^2 \right\} \\ & \leq C \{ \mathcal{E}_N(f^\varepsilon, g^\varepsilon, E^\varepsilon, B^\varepsilon) \}^{\frac{1}{2}} + \mathcal{E}_N(f^\varepsilon, g^\varepsilon, E^\varepsilon, B^\varepsilon) \} \mathcal{D}_N(f^\varepsilon, g^\varepsilon). \end{aligned}$$

Notation: We use \mathcal{E}_N and \mathcal{D}_N instead of $\mathcal{E}_N(f^\varepsilon, g^\varepsilon, E^\varepsilon, B^\varepsilon)$ and $\mathcal{D}_N(f^\varepsilon, g^\varepsilon)$ without confusion.

Proof. Note that (6.1) falls into the category of (5.3) with $\mathcal{A} = 0$. We apply Lemma 5.2 with

$$\begin{aligned} h_1^\varepsilon & \equiv \frac{1}{\varepsilon} \Gamma(f^\varepsilon, f^\varepsilon) + \frac{1}{2} E^\varepsilon \cdot v g^\varepsilon - (E^\varepsilon + v \times B^\varepsilon) \cdot \nabla_v g^\varepsilon, \\ h_2^\varepsilon & \equiv \frac{1}{\varepsilon} \Gamma(g^\varepsilon, f^\varepsilon) + \frac{1}{2} E^\varepsilon \cdot v f^\varepsilon - (E^\varepsilon + v \times B^\varepsilon) \cdot \nabla_v f^\varepsilon, \\ j_1^\varepsilon & \equiv j_2^\varepsilon \equiv 0. \end{aligned}$$

It suffices to estimate the RHS of (5.23): we only need to estimate $\varepsilon^2 \|\partial_\gamma h_\parallel^\varepsilon\|^2$ and $(\partial_\gamma h_1^\varepsilon, \partial_\gamma f^\varepsilon) + (\partial_\gamma h_2^\varepsilon, \partial_\gamma g^\varepsilon)$. Terms related to collision kernel such as $\frac{1}{\varepsilon} \Gamma(f^\varepsilon, f^\varepsilon)$, $\frac{1}{\varepsilon} \Gamma(g^\varepsilon, f^\varepsilon)$ are bounded by $C\{(\mathcal{E}_N)^{\frac{1}{2}} + \mathcal{E}_N\} \mathcal{D}_N$ due to Lemma 7.1 in [9]. Here we show that the ones related to the electromagnetic field also have the same bound. First we look at the projection part $\|\partial_\gamma h_\parallel^\varepsilon\|^2$ including E^ε and B^ε :

$$(6.3) \quad \begin{aligned} & \int_{\mathbb{T}^3} \left[\int_{\mathbb{R}^3} \partial_\gamma \left\{ \frac{1}{2} E^\varepsilon \cdot v g^\varepsilon - (E^\varepsilon + v \times B^\varepsilon) \cdot \nabla_v g^\varepsilon \right\} \zeta dv \right]^2 dx \\ & + \int_{\mathbb{T}^3} \left[\int_{\mathbb{R}^3} \partial_\gamma \left\{ \frac{1}{2} E^\varepsilon \cdot v f^\varepsilon - (E^\varepsilon + v \times B^\varepsilon) \cdot \nabla_v f^\varepsilon \right\} \zeta dv \right]^2 dx. \end{aligned}$$

We will only estimate the first term. Let $\gamma = \gamma_1 + \gamma_2$ where $|\gamma| \leq N - 1$.

$$(I) \equiv \int_{\mathbb{T}^3} \left[\int_{\mathbb{R}^3} \left\{ \frac{1}{2} \partial_{\gamma_1} E^\varepsilon \cdot v \partial_{\gamma_2} g^\varepsilon - (\partial_{\gamma_1} E^\varepsilon + v \times \partial_{\gamma_1} B^\varepsilon) \cdot \nabla_v \partial_{\gamma_2} g^\varepsilon \right\} \zeta dv \right]^2 dx$$

If $|\gamma_1| \leq |\gamma_2|$, we take the sup of $\partial_{\gamma_1} E^\varepsilon, \partial_{\gamma_1} B^\varepsilon$:

$$\begin{aligned} (I) & \leq \int_{\mathbb{T}^3} |\partial_{\gamma_1} E^\varepsilon|^2 \left| \int_{\mathbb{R}^3} v \partial_{\gamma_2} g^\varepsilon \zeta dv \right|^2 dx + \int_{\mathbb{T}^3} |\partial_{\gamma_1} E^\varepsilon|^2 \left| \int_{\mathbb{R}^3} \partial_{\gamma_2} g^\varepsilon \nabla_v \zeta dv \right|^2 dx \\ & \quad + \int_{\mathbb{T}^3} |\partial_{\gamma_1} B^\varepsilon|^2 \left| \int_{\mathbb{R}^3} v \times \partial_{\gamma_2} g^\varepsilon \nabla_v \zeta dv \right|^2 dx \\ & \leq \mathcal{E}_{|\gamma_1|+2} \left\{ \int_{\mathbb{T}^3} \left| \int_{\mathbb{R}^3} v \partial_{\gamma_2} g^\varepsilon \zeta dv \right|^2 dx + \int_{\mathbb{T}^3} \left| \int_{\mathbb{R}^3} \partial_{\gamma_2} g^\varepsilon \nabla_v \zeta dv \right|^2 dx \right. \\ & \quad \left. + \int_{\mathbb{T}^3} \left| \int_{\mathbb{R}^3} v \times \partial_{\gamma_2} g^\varepsilon \nabla_v \zeta dv \right|^2 dx \right\} \quad (\text{Sobolev imbedding theorem}) \\ & \leq \mathcal{E}_{|\gamma_1|+2} \left(\int_{\mathbb{R}^3} |v \zeta|^2 + |\nabla_v \zeta|^2 + |v \nabla_v \zeta|^2 dv \right) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\partial_{\gamma_2} g^\varepsilon|^2 dx dv \quad (\text{H\"older ineq}) \\ & \leq \mathcal{E}_{|\gamma_1|+2} \mathcal{D}_{|\gamma_2|} \quad (\text{Note that } \zeta \text{ decays exponentially}) \\ & \leq \mathcal{E}_N \mathcal{D}_N \quad (N \geq 8); \end{aligned}$$

on the other hand, if $|\gamma_2| < |\gamma_1|$, we take the sup of $\partial_{\gamma_2} g^\varepsilon$:

$$\begin{aligned} (I) & \leq \sup_{x,v} |\partial_{\gamma_2} g^\varepsilon| \left\{ \int_{\mathbb{T}^3} |\partial_{\gamma_1} E^\varepsilon|^2 \left| \int_{\mathbb{R}^3} v \zeta dv \right|^2 dx + \int_{\mathbb{T}^3} |\partial_{\gamma_1} E^\varepsilon|^2 \left| \int_{\mathbb{R}^3} \nabla_v \zeta dv \right|^2 dx \right. \\ & \quad \left. + \int_{\mathbb{T}^3} |\partial_{\gamma_1} B^\varepsilon|^2 \left| \int_{\mathbb{R}^3} v \times \nabla_v \zeta dv \right|^2 dx \right\} \\ & \leq \mathcal{D}_{|\gamma_2|+4} \mathcal{E}_{|\gamma_1|} \\ & \leq \mathcal{E}_N \mathcal{D}_N \quad (N \geq 8). \end{aligned}$$

Thus we conclude that for $|\gamma| \leq N - 1$,

$$(6.4) \quad \varepsilon^2 \|\partial_\gamma h_\parallel^\varepsilon\|^2 \leq C \mathcal{E}_N \mathcal{D}_N.$$

Now we turn to $(\partial_\gamma h_1^\varepsilon, \partial_\gamma f^\varepsilon) + (\partial_\gamma h_2^\varepsilon, \partial_\gamma g^\varepsilon)$ for $|\gamma| \leq N$.

$$\begin{aligned}
& \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \partial_\gamma \left\{ \frac{1}{2} E^\varepsilon \cdot v g^\varepsilon - (E^\varepsilon + v \times B^\varepsilon) \cdot \nabla_v g^\varepsilon \right\} \cdot \partial_\gamma f^\varepsilon dv dx \\
& + \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \partial_\gamma \left\{ \frac{1}{2} E^\varepsilon \cdot v f^\varepsilon - (E^\varepsilon + v \times B^\varepsilon) \cdot \nabla_v f^\varepsilon \right\} \cdot \partial_\gamma g^\varepsilon dv dx \\
(6.5) \quad & = \sum_{\gamma_1 \neq 0} \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \left\{ \frac{1}{2} \partial_{\gamma_1} E^\varepsilon \cdot v \partial_{\gamma_2} g^\varepsilon - (\partial_{\gamma_1} E^\varepsilon + v \times \partial_{\gamma_1} B^\varepsilon) \cdot \nabla_v \partial_{\gamma_2} g^\varepsilon \right\} \cdot \partial_\gamma f^\varepsilon dv dx \\
& + \sum_{\gamma_1 \neq 0} \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \left\{ \frac{1}{2} \partial_{\gamma_1} E^\varepsilon \cdot v \partial_{\gamma_2} f^\varepsilon - (\partial_{\gamma_1} E^\varepsilon + v \times \partial_{\gamma_1} B^\varepsilon) \cdot \nabla_v \partial_{\gamma_2} f^\varepsilon \right\} \cdot \partial_\gamma g^\varepsilon dv dx \\
& + \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} E^\varepsilon \cdot v \partial_\gamma f^\varepsilon \partial_\gamma g^\varepsilon dv dx - \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} (E^\varepsilon + v \times B^\varepsilon) \cdot \nabla_v (\partial_\gamma f^\varepsilon \partial_\gamma g^\varepsilon) dv dx
\end{aligned}$$

The worst last $(|\gamma| + 1)$ -th derivative term of becomes zero after the integration by parts. As we note that $\nu(v) \sim (1 + |v|)$ and $\int \int v f g dv dx \leq C \|f\|_\nu \|g\|_\nu$, we apply the Sobolev imbedding theorem to other terms like the previous argument and then we can get the desired bound $(\mathcal{E}_N)^{\frac{1}{2}} \mathcal{D}_N$. \square

In order to prove Theorem 2.2, it remains to estimate the velocity derivatives.

Proof. (of Theorem 2.2:) We notice that for the hydrodynamic part $[\mathbf{P}_1 f^\varepsilon, \mathbf{P}_2 g^\varepsilon]$,

$$(6.6) \quad \|[\partial_\gamma^\beta \mathbf{P}_1 f^\varepsilon, \partial_\gamma^\beta \mathbf{P}_2 g^\varepsilon]\| \leq C \|[\partial_\gamma \mathbf{P}_1 f^\varepsilon, \partial_\gamma \mathbf{P}_2 g^\varepsilon]\|$$

which has been estimated in Lemma 6.1. It suffices to estimate the remaining microscopic part

$$[\partial_{\gamma_1}^\beta \{\mathbf{I} - \mathbf{P}_1\} f^\varepsilon, \partial_{\gamma_1}^\beta \{\mathbf{I} - \mathbf{P}_2\} g^\varepsilon]$$

for $|\gamma_1| + |\beta| \leq N$, $|\gamma_1| \leq N - 1$ and $\beta > 0$. We take $\partial_{\gamma_1}^\beta$ of equations (2.3) and sum over $|\gamma_1| + |\beta| \leq N$ to get

$$\begin{aligned}
& \partial_t \partial_{\gamma_1}^\beta (\mathbf{I} - \mathbf{P}_1) f^\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x \partial_{\gamma_1}^\beta (\mathbf{I} - \mathbf{P}_1) f^\varepsilon + \frac{1}{\varepsilon^2} \partial_{\gamma_1}^\beta L(\mathbf{I} - \mathbf{P}_1) f^\varepsilon \\
& + \left\{ \partial_t \partial_{\gamma_1}^\beta \mathbf{P}_1 f^\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x \partial_{\gamma_1}^\beta \mathbf{P}_1 f^\varepsilon + \frac{1}{\varepsilon} \binom{\beta_1}{\beta} \partial^{\beta_1} v \cdot \nabla_x \partial_{\gamma_1}^{\beta - \beta_1} f^\varepsilon \right\} \\
& = \partial_{\gamma_1}^\beta \left\{ \frac{1}{\varepsilon} \Gamma(f^\varepsilon, f^\varepsilon) + \frac{1}{2} E^\varepsilon \cdot v g^\varepsilon - (E^\varepsilon + v \times B^\varepsilon) \cdot \nabla_v g^\varepsilon \right\}; \\
(6.7) \quad & \partial_t \partial_{\gamma_1}^\beta (\mathbf{I} - \mathbf{P}_2) g^\varepsilon + \frac{1}{\varepsilon} v \cdot [\nabla_x \partial_{\gamma_1}^\beta (\mathbf{I} - \mathbf{P}_2) g^\varepsilon - \partial^\beta (\sqrt{\mu}) \partial_{\gamma_1} E^\varepsilon] + \frac{1}{\varepsilon^2} \partial_{\gamma_1}^\beta \mathcal{L}(\mathbf{I} - \mathbf{P}_2) g^\varepsilon \\
& + \left\{ \partial_t \partial_{\gamma_1}^\beta \mathbf{P}_2 g^\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x \partial_{\gamma_1}^\beta \mathbf{P}_2 g^\varepsilon + \frac{1}{\varepsilon} \binom{\beta_1}{\beta} \partial^{\beta_1} v \cdot [\nabla_x \partial_{\gamma_1}^{\beta - \beta_1} g^\varepsilon - \partial^{\beta - \beta_1} (\sqrt{\mu}) \partial_{\gamma_1} E^\varepsilon] \right\} \\
& = \partial_{\gamma_1}^\beta \left\{ \frac{1}{\varepsilon} \Gamma(g^\varepsilon, f^\varepsilon) + \frac{1}{2} E^\varepsilon \cdot v f^\varepsilon - (E^\varepsilon + v \times B^\varepsilon) \cdot \nabla_v f^\varepsilon \right\},
\end{aligned}$$

where $|\beta_1| = 1$. We illustrate the estimate on only $\{\mathbf{I} - \mathbf{P}_2\} g^\varepsilon$ and the other one can be done in the same way. Also we will use many results from [9]. Taking the

inner product with $\partial_{\gamma_1}^\beta \{\mathbf{I} - \mathbf{P}_2\} g^\varepsilon$, we get

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \{ \|\partial_{\gamma_1}^\beta (\mathbf{I} - \mathbf{P}_2) g^\varepsilon\|^2 \} + \frac{1}{\varepsilon^2} (\partial_{\gamma_1}^\beta \mathcal{L}(\mathbf{I} - \mathbf{P}_2) g^\varepsilon, \partial_{\gamma_1}^\beta (\mathbf{I} - \mathbf{P}_2) g^\varepsilon) \\
& - \frac{1}{\varepsilon} (v \cdot \partial^\beta (\sqrt{\mu}) \partial_{\gamma_1} E^\varepsilon, \partial_{\gamma_1}^\beta (\mathbf{I} - \mathbf{P}_2) g^\varepsilon) + (\partial_t \partial_{\gamma_1}^\beta \mathbf{P}_2 g^\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x \partial_{\gamma_1}^\beta \mathbf{P}_2 g^\varepsilon) \\
& + \frac{1}{\varepsilon} \left(\frac{\beta_1}{\beta} \right) \partial^{\beta_1} v \cdot \partial^{\beta - \beta_1} \nabla_x \partial_{\gamma_1} g^\varepsilon, \partial_{\gamma_1}^\beta (\mathbf{I} - \mathbf{P}_2) g^\varepsilon) \\
(6.8) \quad & - \left(\frac{1}{\varepsilon} \left(\frac{\beta_1}{\beta} \right) \partial^{\beta_1} v \cdot \partial^{\beta - \beta_1} (\sqrt{\mu}) \partial_{\gamma_1} E^\varepsilon, \partial_{\gamma_1}^\beta (\mathbf{I} - \mathbf{P}_2) g^\varepsilon \right) \\
& = \frac{1}{\varepsilon} (\partial_{\gamma_1}^\beta \Gamma(g^\varepsilon, f^\varepsilon), \partial_{\gamma_1}^\beta (\mathbf{I} - \mathbf{P}_2) g^\varepsilon) \\
& + (\partial_{\gamma_1}^\beta \{ \frac{1}{2} E^\varepsilon \cdot v f^\varepsilon - (E^\varepsilon + v \times B^\varepsilon) \cdot \nabla_v f^\varepsilon \}, \partial_{\gamma_1}^\beta (\mathbf{I} - \mathbf{P}_2) g^\varepsilon).
\end{aligned}$$

Here we only estimate the terms including E^ε or B^ε . We utilize the field estimate from the previous section. Other terms can be estimated in the same way as presented in [9]. See the proof of Theorem 2.2 there (p.38-40). The third term on the LHS of (6.8) can be taken care of as following:

$$\begin{aligned}
& \frac{1}{\varepsilon} (v \cdot \partial^\beta (\sqrt{\mu}) \partial_{\gamma_1} E^\varepsilon, \partial_{\gamma_1}^\beta (\mathbf{I} - \mathbf{P}_2) g^\varepsilon) \\
& \leq C_\xi \|\partial_{\gamma_1} E^\varepsilon\|^2 + \frac{\xi}{\varepsilon^2} \|\partial_{\gamma_1}^\beta (\mathbf{I} - \mathbf{P}_2) g^\varepsilon\|_\nu^2 \quad (\xi \text{ is a small positive fixed number}) \\
& \leq \tilde{C}_\xi \frac{d}{dt} \int_{\mathbb{T}^3} \langle \varepsilon (\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_1} g^\varepsilon, v \sqrt{\mu} \rangle \cdot \partial_{\gamma_1} E^\varepsilon + C_2 \varepsilon \partial_{\gamma_2} E^\varepsilon \cdot \nabla \times \partial_{\gamma_2} B^\varepsilon dx \\
& + \tilde{C}_\xi \{ \|\nabla_x (\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_1} g^\varepsilon\|_\nu^2 + \|(\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_1} g^\varepsilon\|_\nu^2 + \|(\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_2} g^\varepsilon\|_\nu^2 \} + \mathcal{E}_N \mathcal{D}_N \\
& + \frac{\xi}{\varepsilon^2} \|\partial_{\gamma_1}^\beta (\mathbf{I} - \mathbf{P}_2) g^\varepsilon\|_\nu^2 \quad (\text{by (5.21) and (6.4)})
\end{aligned}$$

where $|\gamma_2| \leq N - 2$. One gets the same estimate for $(\frac{1}{\varepsilon} (\frac{\beta_1}{\beta}) \partial^{\beta_1} v \cdot \partial^{\beta - \beta_1} (\sqrt{\mu}) \partial_{\gamma_1} E^\varepsilon, \partial_{\gamma_1}^\beta (\mathbf{I} - \mathbf{P}_2) g^\varepsilon)$, since it is bounded by $C_\xi \|\partial_{\gamma_1} E^\varepsilon\|^2 + \frac{\xi}{\varepsilon^2} \|\partial_{\gamma_1}^\beta (\mathbf{I} - \mathbf{P}_2) g^\varepsilon\|_\nu^2$. Define $\tilde{G}(t)$ by

$$\tilde{G}(t) \equiv \tilde{C}_\xi \int_{\mathbb{T}^3} \langle (\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_1} g^\varepsilon, v \sqrt{\mu} \rangle \cdot \partial_{\gamma_1} E^\varepsilon + C_2 \partial_{\gamma_2} E^\varepsilon \cdot \nabla \times \partial_{\gamma_2} B^\varepsilon dx.$$

Next, the last term in (6.8) will be treated. We decompose f^ε and g^ε into $\mathbf{P}_1 f^\varepsilon + (\mathbf{I} - \mathbf{P}_1) f^\varepsilon$ and $\mathbf{P}_2 g^\varepsilon + (\mathbf{I} - \mathbf{P}_2) g^\varepsilon$ respectively:

$$\begin{aligned}
(6.9) \quad & (\partial_{\gamma_1}^\beta \{ \frac{1}{2} E^\varepsilon \cdot v f^\varepsilon - (E^\varepsilon + v \times B^\varepsilon) \cdot \nabla_v f^\varepsilon \}, \partial_{\gamma_1}^\beta (\mathbf{I} - \mathbf{P}_2) g^\varepsilon) \\
& = (\partial_{\gamma_1}^\beta \{ \frac{1}{2} E^\varepsilon \cdot v \mathbf{P}_1 f^\varepsilon - (E^\varepsilon + v \times B^\varepsilon) \cdot \nabla_v \mathbf{P}_1 f^\varepsilon \}, \partial_{\gamma_1}^\beta (\mathbf{I} - \mathbf{P}_2) g^\varepsilon) \\
& + (\partial_{\gamma_1}^\beta \{ \frac{1}{2} E^\varepsilon \cdot v (\mathbf{I} - \mathbf{P}_1) f^\varepsilon - (E^\varepsilon + v \times B^\varepsilon) \cdot \nabla_v (\mathbf{I} - \mathbf{P}_1) f^\varepsilon \}, \partial_{\gamma_1}^\beta (\mathbf{I} - \mathbf{P}_2) g^\varepsilon) \\
& \equiv (I) + (II)
\end{aligned}$$

The both terms can be computed in the same spirit as in (6.5). The main concern is that the number of derivatives could be $N + 1$ for the worst case due to the Vlasov term $[\nabla_v g^\varepsilon, \nabla_v f^\varepsilon]$. As for (I), we recall that the hydrodynamic parts are

not affected by the velocity derivative as noted in (6.6) and so we can deduce that it is bounded by $(\mathcal{E}_N)^{\frac{1}{2}}\mathcal{D}_N$. For the second term (II) , the $(N+1)$ -th order derivative term together with the similar term stemming from the first equation in (6.7) is gone by the integration by parts:

$$\begin{aligned} & ((E^\varepsilon + v \times B^\varepsilon) \cdot \nabla_v \partial_{\gamma_1}^\beta (\mathbf{I} - \mathbf{P}_1) g^\varepsilon, \partial_{\gamma_1}^\beta (\mathbf{I} - \mathbf{P}_2) f^\varepsilon) \\ & + ((E^\varepsilon + v \times B^\varepsilon) \cdot \nabla_v \partial_{\gamma_1}^\beta (\mathbf{I} - \mathbf{P}_1) f^\varepsilon, \partial_{\gamma_1}^\beta (\mathbf{I} - \mathbf{P}_2) g^\varepsilon) = 0. \end{aligned}$$

and hence it is also bounded by $(\mathcal{E}_N)^{\frac{1}{2}}\mathcal{D}_N$.

Combining the above estimates with Theorem 2.2 in [9], we obtain for $|\gamma_1| \leq N-1$ and $|\beta| + |\gamma_1| = N$,

$$\begin{aligned} & \frac{d}{dt} \left\{ \frac{1}{2} \| [\partial_{\gamma_1}^\beta \{\mathbf{I} - \mathbf{P}_1\} f^\varepsilon, \partial_{\gamma_1}^\beta \{\mathbf{I} - \mathbf{P}_2\} g^\varepsilon] \|^2 \right\} \\ & + \frac{\delta}{4\varepsilon^2} \| [\partial_{\gamma_1}^\beta \{\mathbf{I} - \mathbf{P}_1\} f^\varepsilon, \partial_{\gamma_1}^\beta \{\mathbf{I} - \mathbf{P}_2\} g^\varepsilon] \|^2_\nu \\ & \leq C \{ \| [\partial_{\gamma_1} f^\varepsilon, \partial_{\gamma_1} g^\varepsilon] \|^2_\nu + \frac{1}{\varepsilon^2} (\| \nabla_x (\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_1} g^\varepsilon \|^2_\nu + \| (\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_1} g^\varepsilon \|^2_\nu) \} \\ & + \varepsilon \frac{d}{dt} \tilde{G} + C \{ (\mathcal{E}_N)^{\frac{1}{2}} + \mathcal{E}_N + \varepsilon^2 \} \mathcal{D}_N. \end{aligned}$$

Multiplying the above by factor 4 and adding a large multiple K of (6.2), we get the following for $|\gamma| \leq N$, $|\beta| + |\gamma_1| \leq N$ and $|\gamma_1| \leq N-1$,

$$\begin{aligned} & \frac{d}{dt} [K \{ C_1 \{ \| [\partial_\gamma f^\varepsilon, \partial_\gamma g^\varepsilon] \|^2 + (\| \partial_\gamma E^\varepsilon \|^2 + \| \partial_\gamma B^\varepsilon \|^2) \} - \varepsilon \delta G(t) \} \\ (6.10) \quad & + 2 \| [\partial_{\gamma_1}^\beta \{\mathbf{I} - \mathbf{P}_1\} f^\varepsilon, \partial_{\gamma_1}^\beta \{\mathbf{I} - \mathbf{P}_2\} g^\varepsilon] \|^2 - 4\varepsilon \tilde{G}(t)] + \delta \mathcal{D}_N \\ & \leq C_K \{ (\mathcal{E}_N)^{\frac{1}{2}} + \mathcal{E}_N + \varepsilon^2 \} \mathcal{D}_N. \end{aligned}$$

Notice that

$$\begin{aligned} G(t) & \leq C \sum_{|\gamma| \leq N} \{ \| [\partial_\gamma \mathbf{P}_1 f^\varepsilon, \partial_\gamma \mathbf{P}_2 g^\varepsilon] \| + \| \partial_\gamma E^\varepsilon \| + \| \partial_\gamma B^\varepsilon \| \} \cdot \{ \| [\partial_\gamma \mathbf{P}_1 f^\varepsilon, \partial_\gamma \mathbf{P}_2 g^\varepsilon] \| \\ (6.11) \quad & + \| [\partial_\gamma \{\mathbf{I} - \mathbf{P}_1\} f^\varepsilon, \partial_\gamma \{\mathbf{I} - \mathbf{P}_2\} g^\varepsilon] \| + \| \partial_\gamma E^\varepsilon \| + \| \partial_\gamma B^\varepsilon \| \}. \end{aligned}$$

And $\tilde{G}(t)$ has the same bound. We thus can redefine an instant energy by

$$\begin{aligned} \mathcal{E}_N(f^\varepsilon, g^\varepsilon, E^\varepsilon, B^\varepsilon) & \equiv K \{ C_1 \{ \| [\partial_\gamma f^\varepsilon, \partial_\gamma g^\varepsilon] \|^2 + (\| \partial_\gamma E^\varepsilon \|^2 + \| \partial_\gamma B^\varepsilon \|^2) \} - \varepsilon \delta G(t) \} \\ (6.12) \quad & + 2 \| [\partial_{\gamma_1}^\beta \{\mathbf{I} - \mathbf{P}_1\} f^\varepsilon, \partial_{\gamma_1}^\beta \{\mathbf{I} - \mathbf{P}_2\} g^\varepsilon] \|^2 - 4\varepsilon \tilde{G}(t), \end{aligned}$$

for ε sufficiently small. With such a small $\varepsilon > 0$, adjusting the constants in (6.10) and applying a standard continuity argument, we thus deduce (2.6) by letting \mathcal{E}_N sufficiently small initially. Indeed, our argument is still valid for $\varepsilon = 1$, since we can choose K, C_1 large enough. And that guarantees the global existence of the Vlasov-Maxwell-Boltzmann system.

Since no estimates for the highest N -th derivatives of $E^\varepsilon, B^\varepsilon$ are not available as we can see that full (up to N -th) derivatives of g^ε are needed in the RHS of (5.21) to control the $(N-1)$ -th derivative of E^ε , and hence the dissipation rate is weaker than the instant energy, we do not expect exponential decay unlike the pure Boltzmann case. To obtain a decay rate (2.8), we use an interpolation argument presented in [16]. We point out that our method is slightly easier because we do not have to deal with the time derivative. The key idea is to get a bound for the N -th derivatives $\partial_\gamma E^\varepsilon$ and $\partial_\gamma B^\varepsilon$ with the bound of higher energy. Let k be a

positive integer and $|\gamma| \leq N$. By an interpolation between Sobolev spaces H^{N-1} and H^{N+k} , we get

$$\|\partial_\gamma E^\varepsilon\|^2 + \|\partial_\gamma B^\varepsilon\|^2 \leq C\{\|E^\varepsilon\|_{H^{N-1}}^{\frac{2k}{k+1}} + \|B^\varepsilon\|_{H^{N-1}}^{\frac{2k}{k+1}}\}\{\|E^\varepsilon\|_{H^{N+k}}^{\frac{2}{k+1}} + \|B^\varepsilon\|_{H^{N+k}}^{\frac{2}{k+1}}\}.$$

Due to (2.7), (5.20) and (5.21) we have

$$\begin{aligned} \|E^\varepsilon\|_{H^N}^{\frac{2k+2}{k}} + \|B^\varepsilon\|_{H^N}^{\frac{2k+2}{k}} &\leq C_{N+k}\mathcal{E}_{N+k}(0)^{\frac{1}{k}}\{\|E^\varepsilon\|_{H^{N-1}}^2 + \|B^\varepsilon\|_{H^{N-1}}^2\} \\ &\leq C_{N+k}\mathcal{E}_{N+k}(0)^{\frac{1}{k}}\{\varepsilon \frac{d}{dt}\tilde{G} + \mathcal{D}_N\}, \end{aligned}$$

where ε sufficiently small and

$$(6.13) \quad \tilde{G}(t) \equiv \int_{\mathbb{T}^3} \langle (\mathbf{I} - \mathbf{P}_2) \partial_{\gamma_1} g^\varepsilon, v\sqrt{\mu} \rangle \cdot \partial_{\gamma_1} E^\varepsilon + \partial_{\gamma_2} E^\varepsilon \cdot \nabla \times \partial_{\gamma_2} B^\varepsilon dx,$$

up to constant multiple. In particular, we have $|\tilde{G}(t)| \leq \mathcal{E}_N$. Noting that the other part of \mathcal{E}_N , i.e. the nonelectromagnetic part is bounded by \mathcal{D}_N , a lower bound for \mathcal{D}_N can be given as following:

$$C_{N,k}\mathcal{E}_{N+k}(0)^{-\frac{1}{k}}\mathcal{E}_N^{\frac{k+1}{k}} - \varepsilon \frac{d}{dt}\tilde{G} \leq \mathcal{D}_N.$$

It follows that

$$(6.14) \quad \frac{d}{dt}\{\mathcal{E}_N - \varepsilon\tilde{G}\} + C_{N,k}\mathcal{E}_{N+k}(0)^{-\frac{1}{k}}\mathcal{E}_N^{\frac{k+1}{k}} \leq 0.$$

Letting $\tilde{\mathcal{E}}_N = \mathcal{E}_N - \varepsilon\tilde{G}$, we get $\frac{1}{C}\mathcal{E}_N \leq \tilde{\mathcal{E}}_N \leq C\mathcal{E}_N$ for some $C > 1$ and thus (6.14) becomes

$$\frac{d}{dt}\tilde{\mathcal{E}}_N + C_{N,k}\mathcal{E}_{N+k}(0)^{-\frac{1}{k}}\tilde{\mathcal{E}}_N^{\frac{k+1}{k}} \leq 0.$$

After dividing this by $\tilde{\mathcal{E}}_N^{\frac{k+1}{k}}$ and integrating it over $[0, t]$, we get (2.8). \square

7. HIGH ORDER REMAINDER

In this section, we establish Theorem 2.3 for $n \geq 2$. As noted in [9], the remainder equations (1.16) and (1.17) contain singular terms such as zeroth order terms or first order terms in ε which make it hard to apply the energy estimate directly. The difficulty is resolved by introducing new unknowns $f_R^\varepsilon, g_R^\varepsilon, E_R^\varepsilon$ and B_R^ε , which can be obtained from further construction of $(n+1)$ -th coefficients $f_{n+1}, g_{n+1}, E_{n+1}, B_{n+1}$ and $(n+2)$ -th coefficients $f_{n+2}, g_{n+2}, E_{n+2}, B_{n+2}$ in the diffusive expansion (1.9). The compatibility conditions (1.14) and (1.15) will then eliminate such severe singularity in ε .

We reformulate the problem. Given $f_1, \dots, f_n; g_1, \dots, g_n; E_1, \dots, E_n; B_1, \dots, B_n$ determined by the initial data, we further construct artificial $f_{n+1}, g_{n+1}, E_{n+1}, B_{n+1}$ and $f_{n+2}, g_{n+2}, E_{n+2}, B_{n+2}$ by letting $m = n+1$ and $m = n+2$ in Theorem 2.1 with constant initial data satisfying

$$u_m^0 \equiv -\frac{1}{|\mathbb{T}^3|} \sum_{\substack{i+j=m \\ i,j \geq 1}} \int_{\mathbb{T}^3} E_i^0(x) \times B_j^0(x) dx; \quad \sigma_m^0 \equiv 0;$$

$$\theta_m^0 \equiv -\frac{1}{3|\mathbb{T}^3|} \sum_{\substack{i+j=m \\ i,j \geq 1}} \int_{\mathbb{T}^3} E_i^0(x) \cdot E_j^0(x) + B_i^0(x) \cdot B_j^0(x) dx.$$

Let us introduce new unknowns f_R^ε , g_R^ε , E_R^ε and B_R^ε such that

$$(7.1) \quad \begin{aligned} f_R^\varepsilon &\equiv f_n^\varepsilon - f_n - \varepsilon f_{n+1} - \varepsilon^2 f_{n+2}, & E_R^\varepsilon &\equiv E_n^\varepsilon - E_n - \varepsilon E_{n+1} - \varepsilon^2 E_{n+2}, \\ g_R^\varepsilon &\equiv g_n^\varepsilon - g_n - \varepsilon g_{n+1} - \varepsilon^2 g_{n+2}, & B_R^\varepsilon &\equiv B_n^\varepsilon - B_n - \varepsilon B_{n+1} - \varepsilon^2 B_{n+2}. \end{aligned}$$

Plugging (7.1) into the remainder equations (1.16), (1.17) and applying (1.14) up to $m = n$ as well as (1.36), (1.37) for $m = n + 2$ we get for $n \geq 2$,

$$(7.2) \quad \begin{aligned} \partial_t f_R^\varepsilon + \frac{v}{\varepsilon} \cdot \nabla_x f_R^\varepsilon + \frac{1}{\varepsilon^2} L f_R^\varepsilon &\equiv h_1^\varepsilon, \\ \partial_t g_R^\varepsilon + \frac{v}{\varepsilon} \cdot \nabla_x (g_R^\varepsilon - \sqrt{\mu} E_R^\varepsilon) + \frac{1}{\varepsilon^2} \mathcal{L} g_R^\varepsilon &\equiv h_2^\varepsilon, \end{aligned}$$

$$(7.3) \quad \begin{aligned} \varepsilon \partial_t E_R^\varepsilon - \nabla \times B_R^\varepsilon &= - \int_{\mathbb{R}^3} v g_R^\varepsilon \sqrt{\mu} dv - \varepsilon^3 \partial_t E_{n+2}, & \nabla \cdot B_R^\varepsilon &= 0, \\ \varepsilon \partial_t B_R^\varepsilon + \nabla \times E_R^\varepsilon &= - \varepsilon^3 \partial_t B_{n+2}, & \nabla \cdot E_R^\varepsilon &= \int_{\mathbb{R}^3} g_R^\varepsilon \sqrt{\mu} dv, \end{aligned}$$

where

$$\begin{aligned} h_1^\varepsilon &\equiv h_1(f_R^\varepsilon) + h_1(f) + h_1(E_R^\varepsilon, B_R^\varepsilon) + h_1(E, B), \\ h_2^\varepsilon &\equiv h_2(g_R^\varepsilon) + h_2(g) + h_2(E_R^\varepsilon, B_R^\varepsilon) + h_2(E, B), \end{aligned}$$

and each h_i is given as follows:

$$(7.4) \quad \begin{aligned} h_1(f_R^\varepsilon) &\equiv \varepsilon^{n-2} \Gamma(f_R^\varepsilon, f_R^\varepsilon) + \sum_{i=1}^{n+2} \varepsilon^{i-2} \{\Gamma(f_R^\varepsilon, f_i) + \Gamma(f_i, f_R^\varepsilon)\}, \\ h_1(f) &\equiv \sum_{i+j \geq n+3} \varepsilon^{i+j-n-2} \Gamma(f_i, f_j) - \varepsilon \{\partial_t f_{n+1} + v \cdot \nabla_x f_{n+2}\} - \varepsilon^2 \partial_t f_{n+2}, \\ h_1(E_R^\varepsilon, B_R^\varepsilon) &\equiv -\frac{\varepsilon^{n-1}}{\sqrt{\mu}} (E_R^\varepsilon + v \times B_R^\varepsilon) \cdot \nabla_v (g_R^\varepsilon \sqrt{\mu}) \\ &\quad - \frac{1}{\sqrt{\mu}} \sum_{i=1}^{n+2} \varepsilon^{i-1} \{(E_i + v \times B_i) \cdot \nabla_v (g_R^\varepsilon \sqrt{\mu}) + (E_R^\varepsilon + v \times B_R^\varepsilon) \cdot \nabla_v (g_i \sqrt{\mu})\}, \\ h_1(E, B) &\equiv -\frac{1}{\sqrt{\mu}} \sum_{i+j \geq n+2} \varepsilon^{i+j-n-1} (E_i + v \times B_i) \cdot \nabla_v (g_j \sqrt{\mu}), \\ h_2(g_R^\varepsilon) &\equiv \varepsilon^{n-2} \Gamma(g_R^\varepsilon, f_R^\varepsilon) + \sum_{i=1}^{n+2} \varepsilon^{i-2} \{\Gamma(g_R^\varepsilon, f_i) + \Gamma(g_i, f_R^\varepsilon)\}, \\ h_2(g) &\equiv \sum_{i+j \geq n+3} \varepsilon^{i+j-n-2} \Gamma(g_i, f_j) - \varepsilon \{\partial_t g_{n+1} + v \cdot \nabla_x g_{n+2}\} - \varepsilon^2 \partial_t g_{n+2}, \\ h_2(E_R^\varepsilon, B_R^\varepsilon) &\equiv -\frac{\varepsilon^{n-1}}{\sqrt{\mu}} (E_R^\varepsilon + v \times B_R^\varepsilon) \cdot \nabla_v (f_R^\varepsilon \sqrt{\mu}) \\ &\quad - \frac{1}{\sqrt{\mu}} \sum_{i=1}^{n+2} \varepsilon^{i-1} \{(E_i + v \times B_i) \cdot \nabla_v (f_R^\varepsilon \sqrt{\mu}) + (E_R^\varepsilon + v \times B_R^\varepsilon) \cdot \nabla_v (f_i \sqrt{\mu})\}, \end{aligned}$$

$$h_2(E, B) \equiv -\frac{1}{\sqrt{\mu}} \sum_{i+j \geq n+2} \varepsilon^{i+j-n-1} (E_i + v \times B_i) \cdot \nabla_v (f_j \sqrt{\mu}) + \varepsilon E_{n+2} \cdot v \sqrt{\mu}.$$

Our goal is to study f_R^ε , g_R^ε , E_R^ε and B_R^ε which are equivalent to f_n^ε , g_n^ε , E_n^ε and B_n^ε via (7.1). The procedure closely follows the proofs of the last section. We first establish pure spatial energy estimate in Lemma 7.1. We write the hydrodynamic field parts $\mathbf{P}_1 f_R^\varepsilon$ and $\mathbf{P}_2 g_R^\varepsilon$ as:

$$[\mathbf{P}_1 f_R^\varepsilon, \mathbf{P}_2 g_R^\varepsilon] = [\{a_R^\varepsilon(t, x) + b_R^\varepsilon(t, x) \cdot v + c_R^\varepsilon(t, x)|v|^2\} \sqrt{\mu}, d_R^\varepsilon(t, x) \sqrt{\mu}].$$

Lemma 7.1. *Assume that (1.8) are the solution to the kinetic equation (1.6) such that the following conservation laws hold:*

$$\begin{aligned}
(7.5) \quad \int_{\mathbb{T}^3} a_R^\varepsilon dx &= \frac{\varepsilon^n}{2} \int_{\mathbb{T}^3} |E_R^\varepsilon|^2 + |B_R^\varepsilon|^2 dx + \sum_{i=1}^{n+2} \varepsilon^i \int_{\mathbb{T}^3} \{E_i \cdot E_R^\varepsilon + B_i \cdot B_R^\varepsilon\} dx \\
&\quad + \frac{1}{2} \sum_{\substack{i+j \geq n+3 \\ 1 \leq i, j \leq n+2}} \varepsilon^{i+j-n} \int_{\mathbb{T}^3} E_i \cdot E_j + B_i \cdot B_j dx, \\
\int_{\mathbb{T}^3} b_R^\varepsilon dx &= -\varepsilon^n \int_{\mathbb{T}^3} E_R^\varepsilon \times B_R^\varepsilon dx - \sum_{i=1}^{n+2} \varepsilon^i \int_{\mathbb{T}^3} \{E_i \times B_R^\varepsilon + E_R^\varepsilon \times B_i\} dx \\
&\quad - \sum_{\substack{i+j \geq n+3 \\ 1 \leq i, j \leq n+2}} \varepsilon^{i+j-n} \int_{\mathbb{T}^3} E_i \times B_j dx, \\
\int_{\mathbb{T}^3} c_R^\varepsilon dx &= -\frac{\varepsilon^n}{6} \int_{\mathbb{T}^3} |E_R^\varepsilon|^2 + |B_R^\varepsilon|^2 dx - \frac{1}{3} \sum_{i=1}^{n+2} \varepsilon^i \int_{\mathbb{T}^3} \{E_i \cdot E_R^\varepsilon + B_i \cdot B_R^\varepsilon\} dx \\
&\quad - \frac{1}{6} \sum_{\substack{i+j \geq n+3 \\ 1 \leq i, j \leq n+2}} \varepsilon^{i+j-n} \int_{\mathbb{T}^3} E_i \cdot E_j + B_i \cdot B_j dx, \\
\int_{\mathbb{T}^3} d_R^\varepsilon dx &= 0 \quad \text{and} \quad \int_{\mathbb{T}^3} B_R^\varepsilon dx = 0.
\end{aligned}$$

Then there is $\lambda > 0$ so that for any $\xi > 0$, there exists some polynomial $U_\xi(0) = 0$ such that

$$\begin{aligned}
(7.6) \quad &\frac{d}{dt} \{C_1 \sum_{|\gamma| \leq N} \{||[\partial_\gamma f_R^\varepsilon, \partial_\gamma g_R^\varepsilon]||^2 + ||[\partial_\gamma E_R^\varepsilon, \partial_\gamma B_R^\varepsilon]||^2\} - \varepsilon \delta G(t)\} \\
&+ \delta \sum_{|\gamma| \leq N} \left\{ \frac{1}{\varepsilon^2} ||[\partial_\gamma \{\mathbf{I} - \mathbf{P}_1\} f_R^\varepsilon, \partial_\gamma \{\mathbf{I} - \mathbf{P}_2\} g_R^\varepsilon]||_\nu^2 + ||[\partial_\gamma \mathbf{P}_1 f_R^\varepsilon, \partial_\gamma \mathbf{P}_2 g_R^\varepsilon]||^2 \right\} \\
&\leq e^{-\lambda t} U_\xi(||\{\mathbf{u}_n, \theta_n, \sigma_n\}||_N^2) \{\varepsilon^2 + \mathcal{E}_N\} \\
&\quad + C\{(\mathcal{E}_N)^{\frac{1}{2}} + \mathcal{E}_N + \varepsilon U_\xi(||\{\mathbf{u}_n, \theta_n, \sigma_n\}||_N) + \xi\} \mathcal{D}_N,
\end{aligned}$$

for ε sufficiently small, $\mathcal{E}_N \equiv \mathcal{E}_N(f_R^\varepsilon, g_R^\varepsilon, E_R^\varepsilon, B_R^\varepsilon)(t)$ and $\mathcal{D}_N \equiv \mathcal{D}_N(f_R^\varepsilon, g_R^\varepsilon)(t)$.

Notation: $\mathcal{E}_N[i] \equiv \mathcal{E}_N(f_i, g_i, E_i, B_i)(t)$.

Proof. Note that (7.5) falls into the category of (5.3) with

$$\mathcal{A} \equiv \sum_{i=1}^{n+2} (\|E_i\|^2 + \|B_i\|^2).$$

We apply Lemma 5.1 and Lemma 5.2 with $j_1^\varepsilon = -\varepsilon^3 \partial_t E_{n+2}$, $j_2^\varepsilon = -\varepsilon^3 \partial_t B_{n+2}$ to obtain the following, an analogue of (5.23):

$$\begin{aligned} (7.7) \quad & \frac{d}{dt} \{C_1 \sum_{|\gamma| \leq N} \{ \|\partial_\gamma f_R^\varepsilon, \partial_\gamma g_R^\varepsilon\|^2 + \|\partial_\gamma E_R^\varepsilon, \partial_\gamma B_R^\varepsilon\|^2 \} - \varepsilon \delta G(t) \} \\ & + \delta \sum_{|\gamma| \leq N} \{ \frac{1}{\varepsilon^2} \|\partial_\gamma \{\mathbf{I} - \mathbf{P}_1\} f_R^\varepsilon, \partial_\gamma \{\mathbf{I} - \mathbf{P}_2\} g_R^\varepsilon\|_\nu^2 + \|\partial_\gamma \mathbf{P}_1 f_R^\varepsilon, \partial_\gamma \mathbf{P}_2 g_R^\varepsilon\|^2 \} \\ & \leq 2C_1 \sum_{|\gamma| \leq N} \{ (\partial_\gamma h_1^\varepsilon, \partial_\gamma f_R^\varepsilon) + (\partial_\gamma h_2^\varepsilon, \partial_\gamma g_R^\varepsilon) \} + \varepsilon^2 \delta \sum_{|\gamma| \leq N-1} \|\partial_\gamma h_\parallel^\varepsilon\|^2 \\ & + 2C_1 \varepsilon^2 (\|\partial_\gamma E_R^\varepsilon\| \cdot \|\partial_t \partial_\gamma E_{n+2}\| + \|\partial_\gamma B_R^\varepsilon\| \cdot \|\partial_t \partial_\gamma B_{n+2}\|) \\ & + \varepsilon^2 \delta \sum_{i=1}^{n+2} (\|E_i\|^2 + \|B_i\|^2) + \varepsilon^8 \delta (\|\partial_t E_{n+2}\|^2 + \|\partial_t B_{n+2}\|^2) \\ & \equiv (I) + (II) + (III). \end{aligned}$$

Now it suffices to estimate (I), (II) and (III) in the above to finish Lemma. Observe that by Theorem 2.1

$$(7.8) \quad (III) \leq C \varepsilon^2 e^{-2\lambda t} U(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N^2).$$

As for (II), we use the Cauchy-Schwartz inequality:

$$\begin{aligned} (7.9) \quad (II) & \leq C \varepsilon^2 (\mathcal{E}_N)^{\frac{1}{2}} e^{-\lambda t} U(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N) \\ & \leq C e^{-\lambda t} U(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N) \{\varepsilon^4 + \mathcal{E}_N\}. \end{aligned}$$

The estimate for (I) can be done similarly as in the pure Boltzmann case. Electromagnetic related terms do not cause any technical difficulty. Following the proof of Lemma 8.1 in [9], we can obtain

$$\begin{aligned} (7.10) \quad & \varepsilon^2 \sum_{|\gamma| \leq N-1} \|\partial_\gamma h_\parallel^\varepsilon\|^2 \leq e^{-2\lambda t} U(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N^2) \{\varepsilon^2 + \mathcal{E}_N\} \\ & + C \{\varepsilon^2 \mathcal{E}_N + \varepsilon^2 e^{-2\lambda t} U(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N^2)\} \mathcal{D}_N; \\ & \sum_{|\gamma| \leq N} \{ (\partial_\gamma h_1^\varepsilon, \partial_\gamma f_R^\varepsilon) \} + (\partial_\gamma h_2^\varepsilon, \partial_\gamma g_R^\varepsilon) \leq e^{-\lambda t} U_\xi(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N^2) \{\varepsilon^2 + \mathcal{E}_N\} \\ & + C \{\varepsilon (\mathcal{E}_N)^{\frac{1}{2}} + \varepsilon U_\xi(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N) + \xi\} \mathcal{D}_N. \end{aligned}$$

Hence, as adjusting constants and collecting terms, we deduce our lemma. \square

In order to prove Theorem 2.3, it remains to estimate the velocity derivatives.

Proof. (of Theorem 2.3:) It suffices to estimate just the remaining part

$$[\partial_\gamma^\beta \{\mathbf{I} - \mathbf{P}_1\} f_R^\varepsilon, \partial_\gamma^\beta \{\mathbf{I} - \mathbf{P}_2\} g_R^\varepsilon].$$

We follow exactly the same argument in the proof of Theorem 2.2. Comparing $\partial_{\gamma_1}^\beta$ derivative of (7.2) with (6.7), we notice the only difference is that now $\partial_{\gamma_1}^\beta h_1^\varepsilon$ and $\partial_{\gamma_1}^\beta h_2^\varepsilon$ are more complicated. Thus it is enough to estimate only

$$(\partial_{\gamma_1}^\beta h_1^\varepsilon, \partial_{\gamma_1}^\beta \{\mathbf{I} - \mathbf{P}_1\} f_R^\varepsilon) + (\partial_{\gamma_1}^\beta h_2^\varepsilon, \partial_{\gamma_1}^\beta \{\mathbf{I} - \mathbf{P}_2\} g_R^\varepsilon).$$

Here we illustrate how it works for electromagnetic related terms. The terms containing $h_1(f), h_1(f_R^\varepsilon), h_2(g)$ or $h_2(g_R^\varepsilon)$ can be done in the same way as in [9]. See the proof of Theorem 2.3 there (p.50-51). Recalling (2.2), we first get

$$\begin{aligned} & (\partial_{\gamma_1}^\beta h_1(E, B), \partial_{\gamma_1}^\beta \{\mathbf{I} - \mathbf{P}_1\} f_R^\varepsilon) + (\partial_{\gamma_1}^\beta h_2(E, B), \partial_{\gamma_1}^\beta \{\mathbf{I} - \mathbf{P}_2\} g_R^\varepsilon) \\ & \leq \varepsilon e^{-\lambda t} U(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N) \|[\partial_{\gamma_1}^\beta \{\mathbf{I} - \mathbf{P}_1\} f_R^\varepsilon, \partial_{\gamma_1}^\beta \{\mathbf{I} - \mathbf{P}_2\} g_R^\varepsilon]\|_\nu \\ & \leq \varepsilon^2 e^{-\lambda t} U(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N) (\mathcal{D}_N)^{\frac{1}{2}} \\ & \leq \varepsilon^2 e^{-2\lambda t} U_\xi(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N^2) + \frac{\xi}{2} \mathcal{D}_N. \end{aligned}$$

Note that we have also taken into account $\varepsilon \partial_{\gamma_1}^\beta (E_{n+2} \cdot v \sqrt{\mu})$ in $h_2(E, B)$. On the other hand, as for $h_1(E_R^\varepsilon, B_R^\varepsilon)$ and $h_2(E_R^\varepsilon, B_R^\varepsilon)$, by splitting $f_R^\varepsilon, g_R^\varepsilon$ into $\mathbf{P}_1 f_R^\varepsilon + (\mathbf{I} - \mathbf{P}_1) f_R^\varepsilon, \mathbf{P}_2 g_R^\varepsilon + (\mathbf{I} - \mathbf{P}_2) g_R^\varepsilon$ as in (6.9), one can get

$$\begin{aligned} & (\partial_{\gamma_1}^\beta h_1(E_R^\varepsilon, B_R^\varepsilon), \partial_{\gamma_1}^\beta \{\mathbf{I} - \mathbf{P}_1\} f_R^\varepsilon) + (\partial_{\gamma_1}^\beta h_2(E_R^\varepsilon, B_R^\varepsilon), \partial_{\gamma_1}^\beta \{\mathbf{I} - \mathbf{P}_2\} g_R^\varepsilon) \\ & \leq \varepsilon (\mathcal{E}_N)^{\frac{1}{2}} \|[\partial_{\gamma_1}^\beta g_R^\varepsilon, \partial_{\gamma_1}^\beta f_R^\varepsilon]\|_\nu \cdot \|[\partial_{\gamma_1}^\beta \{\mathbf{I} - \mathbf{P}_1\} f_R^\varepsilon, \partial_{\gamma_1}^\beta \{\mathbf{I} - \mathbf{P}_2\} g_R^\varepsilon]\|_\nu \\ & \quad + \varepsilon \sum_{i=1}^{n+2} \varepsilon^{i-1} \{(\mathcal{E}_N[i])^{\frac{1}{2}} \|[\partial_{\gamma_1}^\beta g_R^\varepsilon, \partial_{\gamma_1}^\beta f_R^\varepsilon]\|_\nu + (\mathcal{E}_N)^{\frac{1}{2}} \|[\partial_{\gamma_1}^\beta \nabla_v g_i, \partial_{\gamma_1}^\beta \nabla_v f_i]\|_\nu\} \\ & \quad \cdot \frac{1}{\varepsilon} \|[\partial_{\gamma_1}^\beta \{\mathbf{I} - \mathbf{P}_1\} f_R^\varepsilon, \partial_{\gamma_1}^\beta \{\mathbf{I} - \mathbf{P}_2\} g_R^\varepsilon]\|_\nu \\ & \leq \varepsilon (\mathcal{E}_N)^{\frac{1}{2}} \mathcal{D}_N + \varepsilon e^{-\lambda t} U(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N) \{\mathcal{D}_N + (\mathcal{E}_N)^{\frac{1}{2}} (\mathcal{D}_N)^{\frac{1}{2}}\} \\ & \leq \varepsilon e^{-\lambda t} U(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N) \mathcal{E}_N + \{\varepsilon (\mathcal{E}_N)^{\frac{1}{2}} + e^{-\lambda t} U(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N)\} \mathcal{D}_N. \end{aligned}$$

Therefore, as in the proof of Theorem 2.2, with all the above estimates as well as the ones in [9] (p.50-51), we deduce that for any $\xi > 0$,

$$\begin{aligned} & \frac{d}{dt} [K \{C_1 \{ \|[\partial_\gamma f_R^\varepsilon, \partial_\gamma g_R^\varepsilon]\|^2 + \|[\partial_\gamma E_R^\varepsilon, \partial_\gamma B_R^\varepsilon]\|^2 \} - \varepsilon \delta G(t) \} \\ & \quad + 2\delta \|[\partial_\gamma \{\mathbf{I} - \mathbf{P}_1\} f_R^\varepsilon, \partial_\gamma \{\mathbf{I} - \mathbf{P}_2\} g_R^\varepsilon]\|^2 - 4\varepsilon \delta \tilde{G}(t)] + \delta \mathcal{D}_N \\ (7.11) \quad & \leq e^{-\lambda t} U_\xi(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N^2) \{\varepsilon^2 + \mathcal{E}_N\} \\ & \quad + C \{(\mathcal{E}_N)^{\frac{1}{2}} + \mathcal{E}_N + \varepsilon U_\xi(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N) + \xi\} \mathcal{D}_N. \end{aligned}$$

We redefine an equivalent instant energy functional \mathcal{E}_N as in (6.12) to get

$$\begin{aligned} \frac{d}{dt} \mathcal{E}_N + \mathcal{D}_N & \leq e^{-\lambda t} U_\xi(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N^2) \{\varepsilon^2 + \mathcal{E}_N\} \\ & \quad + C \{(\mathcal{E}_N)^{\frac{1}{2}} + \mathcal{E}_N + \varepsilon U_\xi(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N) + \xi\} \mathcal{D}_N. \end{aligned}$$

Choose and then fix ε and ξ so that $\varepsilon U_\xi(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N)$ is sufficiently small. We also assume that

$$\mathcal{E}_N \leq M$$

sufficiently small such that the coefficient in front of \mathcal{D}_N satisfies

$$C\{(\mathcal{E}_N)^{\frac{1}{2}} + \mathcal{E}_N + \varepsilon U_\xi(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N) + \xi\} < \frac{1}{2}.$$

Thus we obtain

$$(7.12) \quad \frac{d}{dt}\mathcal{E}_N + \frac{1}{2}\mathcal{D}_N \leq C e^{-\lambda t} U(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N^2) \{\varepsilon^2 + \mathcal{E}_N\}.$$

In turn, we have

$$\begin{aligned} & \frac{d}{dt} \{e^{-CU(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N^2)} \int_0^t e^{-\lambda s} ds \mathcal{E}_N\} \\ & \leq C e^{-CU(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N^2)} \int_0^t e^{-\lambda s} ds e^{-\lambda t} \varepsilon^2 U(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N^2). \end{aligned}$$

Integrating over t , we deduce

$$\sup_{0 \leq t \leq \infty} \mathcal{E}_N(t) \leq e^{CU(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N^2)} \int_0^\infty e^{-\lambda s} ds \{\mathcal{E}_N(0) + C \varepsilon^2 U(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N^2)\}.$$

We conclude for ε sufficiently small and for some other polynomial U ,

$$(7.13) \quad \sup_{0 \leq t \leq \infty} \mathcal{E}_N(t) \leq e^{U(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N^2)} \{\mathcal{E}_N(0) + \varepsilon^2 U(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N)\} < M/2.$$

A standard continuity argument shows that the hypothesis $\mathcal{E}_N(t) \leq M$ is valid and (7.13) is proven.

Recalling

$$\begin{aligned} f_R^\varepsilon &= \{f_n^\varepsilon - f_n\} + \varepsilon f_{n+1} + \varepsilon^2 f_{n+2}, \quad E_R^\varepsilon = \{E_n^\varepsilon - E_n\} + \varepsilon E_{n+1} + \varepsilon^2 E_{n+2}, \\ g_R^\varepsilon &= \{g_n^\varepsilon - g_n\} + \varepsilon g_{n+1} + \varepsilon^2 g_{n+2}, \quad B_R^\varepsilon = \{B_n^\varepsilon - B_n\} + \varepsilon B_{n+1} + \varepsilon^2 B_{n+2}, \end{aligned}$$

and by Theorem 2.1,

$$(7.14) \quad \begin{aligned} & \mathcal{E}_N(\varepsilon f_{n+1} + \varepsilon^2 f_{n+2}, \varepsilon g_{n+1} + \varepsilon^2 g_{n+2}, \varepsilon E_{n+1} + \varepsilon^2 E_{n+2}, \varepsilon B_{n+1} + \varepsilon^2 B_{n+2}) \\ & \leq C \varepsilon^2 e^{-2\lambda t} U(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N^2), \end{aligned}$$

we thus deduce (2.12) for $f_n^\varepsilon - f_n$, $g_n^\varepsilon - g_n$, $E_n^\varepsilon - E_n$, $B_n^\varepsilon - B_n$.

To get a decay rate (2.13), we use the interpolation argument as done in the last part of the previous section. Let $|\gamma| \leq N$ and $k \geq 1$. Recall that

$$\|\partial_\gamma E_R^\varepsilon\|^2 + \|\partial_\gamma B_R^\varepsilon\|^2 \leq C \{\|E_R^\varepsilon\|_{H^{N-1}}^{\frac{2k}{k+1}} + \|B_R^\varepsilon\|_{H^{N-1}}^{\frac{2k}{k+1}}\} \{\|E_R^\varepsilon\|_{H^{N+k}}^{\frac{2}{k+1}} + \|B_R^\varepsilon\|_{H^{N+k}}^{\frac{2}{k+1}}\}.$$

Denote a bound for \mathcal{E}_N by I_N , i.e.

$$I_N \equiv e^{U(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N^2)} \{\mathcal{E}_N(0) + \varepsilon^2 U(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N)\}.$$

Due to (5.20), (5.21), (7.10) and (7.13) we have

$$\begin{aligned} \|E_R^\varepsilon\|_{H^N}^{\frac{2k+2}{k}} + \|B_R^\varepsilon\|_{H^N}^{\frac{2k+2}{k}} & \leq C_{N+k} (I_{N+k})^{\frac{1}{k}} \{\|E_R^\varepsilon\|_{H^{N-1}}^2 + \|B_R^\varepsilon\|_{H^{N-1}}^2\} \\ & \leq C_{N+k} (I_{N+k})^{\frac{1}{k}} \{\varepsilon \frac{d}{dt} \tilde{G} + \mathcal{D}_N + e^{-2\lambda t} U(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N^2) I_N\}, \end{aligned}$$

where ε sufficiently small and $\tilde{G}(t)$ is defined in (6.13) so that $|\tilde{G}(t)| \leq \mathcal{E}_N$. Noting that the other part of \mathcal{E}_N , i.e. the nonelectromagnetic part is bounded by \mathcal{D}_N , a lower bound for \mathcal{D}_N can be given as following:

$$C_{N,k} (I_{N+k})^{-\frac{1}{k}} \mathcal{E}_N^{\frac{k+1}{k}} - \varepsilon \frac{d}{dt} \tilde{G} - e^{-2\lambda t} U(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N^2) I_N \leq \mathcal{D}_N.$$

It follows from (7.12) that

$$(7.15) \quad \frac{d}{dt}\{\mathcal{E}_N - \varepsilon \tilde{G}\} + C_{N,k}(I_{N+k})^{-\frac{1}{k}} \tilde{\mathcal{E}}_N^{\frac{k+1}{k}} \leq C e^{-\lambda t} I_N U(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N^2).$$

Letting $\tilde{\mathcal{E}}_N = \mathcal{E}_N - \varepsilon \tilde{G}$, we get $\frac{1}{C} \mathcal{E}_N \leq \tilde{\mathcal{E}}_N \leq C \mathcal{E}_N$ for some $C > 1$ and thus (7.15) becomes

$$\frac{d}{dt} \tilde{\mathcal{E}}_N + C_{N,k}(I_{N+k})^{-\frac{1}{k}} \tilde{\mathcal{E}}_N^{\frac{k+1}{k}} \leq C e^{-\lambda t} I_N U(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N^2).$$

Multiply the above by $(1 + \frac{t}{k})^k$, we get

$$(7.16) \quad \frac{d}{dt} \{(1 + \frac{t}{k})^k \tilde{\mathcal{E}}_N(t)\} \leq Q(t) + C e^{-\lambda t} (1 + \frac{t}{k})^k I_N U(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N^2),$$

where

$$\begin{aligned} Q(t) &\equiv (1 + \frac{t}{k})^{k-1} \tilde{\mathcal{E}}_N(t) - C_{N,k}(I_{N+k})^{-\frac{1}{k}} (1 + \frac{t}{k})^k \tilde{\mathcal{E}}_N^{\frac{k+1}{k}}(t) \\ &= (1 + \frac{t}{k})^{k-1} \tilde{\mathcal{E}}_N(t) \{1 - C_{N,k}(I_{N+k})^{-\frac{1}{k}} \{(1 + \frac{t}{k})^k \tilde{\mathcal{E}}_N(t)\}^{\frac{1}{k}}\}. \end{aligned}$$

To conclude our theorem, it suffices to verify the following statement:

Claim. There exists $\tilde{C}_{N+k} > 0$ such that

$$\sup_t \{(1 + \frac{t}{k})^k \tilde{\mathcal{E}}_N(t)\} \leq \tilde{C}_{N+k} I_{N+k},$$

since the same conclusion is valid for \mathcal{E}_N and recalling (7.1), combining with (7.14), we can deduce (2.13).

Proof of Claim: Let S be the set of t such that $Q(t) < 0$. Note that $Q(t) > 0$ for sufficiently small M and small t , which implies that S^c is nonempty. If S is an empty set, namely $Q(t) \geq 0$ for all t , we can set $\tilde{C}_{N+k} = (C_{N,k})^{-k}$. Let $t_1 \in S$. We can find $t_0 \leq t_1$ so that $t_0 \in S^c$ and $Q(t) \leq 0$ for $t_0 \leq t \leq t_1$. Integrate (7.16) from t_0 to t_1 to get

$$\begin{aligned} (1 + \frac{t_1}{k})^k \tilde{\mathcal{E}}_N(t_1) &\leq (1 + \frac{t_0}{k})^k \tilde{\mathcal{E}}_N(t_0) + C I_N U(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N^2) \int_{t_0}^{t_1} e^{-\lambda s} (1 + \frac{s}{k})^k ds \\ &\leq (C_{N,k})^{-k} I_{N+k} + C I_N U(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N^2) \int_0^\infty e^{-\lambda s} (1 + \frac{s}{k})^k ds. \end{aligned}$$

Choosing $\tilde{C}_{N+k} = (C_{N,k})^{-k} + C U(\|\{\mathbf{u}_n, \theta_n, \sigma_n\}\|_N^2) \int_0^\infty e^{-\lambda s} (1 + \frac{s}{k})^k ds$, we conclude the proof. \square

Acknowledgments: The author would like to deeply thank YAN GUO for many stimulating discussions.

REFERENCES

- [1] C. BARDOS, F. GOLSE, D. LEVERMORE: Fluid dynamic limits of kinetic equations. I Formal derivations, *J. Statist. Phys.* **63**, 323-344 (1991)
- [2] C. BARDOS, F. GOLSE, D. LEVERMORE: Fluid dynamic limits of kinetic equations. II convergence proofs for the Boltzmann equation, *Comm. Pure appl. Math.* **46**, 667-753 (1993)
- [3] C. BARDOS, F. GOLSE, D. LEVERMORE: The acoustic limit for the Boltzmann equation, *Arch. Rational. Mech. Anal.* **153**, 177-204 (2000)

- [4] S. BASTEA, R. ESPOSITO, J. L. LEBOWITZ, R. MARRA: Binary fluids with long range segregating interaction. I: Derivation of kinetic and hydrodynamic equations, *J. Statist. Phys.* **101**, 1087-1136 (2000)
- [5] R. DiPERNA, P.-L. LIONS: On the Cauchy problem for the Boltzmann equations: global existence and weak stability, *Ann. of Math.* **130**, 321-366 (1989)
- [6] F. GOLSE, D. LEVERMORE: Stokes-Fourier and Acoustic limits for the Boltzmann equation: Convergence proofs, *Comm. Pure Appl. Math.* **55**, 336-393 (2002)
- [7] F. GOLSE, L. SAINT-RAYMOND: The Navier-Stokes limit of the Boltzmann equation for bounded collision kernels, *Invent. Math.* **155**, 81-161 (2004)
- [8] Y. GUO: The Vlasov-Maxwell-Boltzmann system near Maxwellians, *Invent. Math.* **153**, 593-630 (2003)
- [9] Y. GUO: Boltzmann diffusive limit beyond the Navier-Stokes approximation, *Comm. Pure Appl. Math.* **58**, 1-62 (2005)
- [10] D. LEVERMORE, N. MASMOUDI: From the Boltzmann equation to an incompressible Navier-Stokes-Fourier system, Preprint
- [11] P.-L. LIONS, N. MASMOUDI: From Boltzmann equations to incompressible fluid mechanics equation. I, *Arch. Rational. Mech. Anal.* **158**, 173-193 (2001)
- [12] P.-L. LIONS, N. MASMOUDI: From Boltzmann equations to incompressible fluid mechanics equation. II, *Arch. Rational. Mech. Anal.* **158**, 195-211 (2001)
- [13] N. MASMOUDI: Hydrodynamic limits of the Boltzmann equation: Recent developments, *Bal. Soc. Esp. Mat. Apl.* **26**, 57-78 (2003)
- [14] N. MASMOUDI, L. SAINT-RAYMOND: From the Boltzmann equation to Stokes-Fourier system in a bounded domain, *Comm. Pure Appl. Math.* **56**, 1263-1293 (2003)
- [15] L. SAINT-RAYMOND: From the BGK model to the Navier-Stokes equations, *Ann. Sci. Ecole. Norm. Sup.* **36**, 271-317 (2003)
- [16] R. STRAIN, Y. GUO: Almost exponential decay near Maxwellians, *Comm. Partial Diff. Equations*. in press (2005)
- [17] S. UKAI, K. ASANO: The Euler limit and initial layer of the nonlinear Boltzmann equation, *Hokkido Math. J.* **12**, 311-332 (1983)
- [18] C. VILLANI: A review of mathematical topics in collisional kinetic theory, *Handbook of mathematical fluid mechanics, Vol.I* 71-305 (2002)

DEPARTMENT OF MATHEMATICS, BROWN UNIVERSITY, PROVIDENCE, RI 02912, USA
 E-mail address: juhijang@math.brown.edu